M321 S1–16

THE OPEN UNIVERSITY

Mathematics: A third Level Course
Partial Differential Equations of Applied Mathematics

Supplementary Material M321 1–16

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Partial Differential Equations of Applied Mathematics

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The Open University

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Contents

Radio Programmes 4 and 8 are Tutorial Programmes.

M321 R1

RADIO PROGRAMME 1

Broadcast Notes

INTRODUCING THE COURSE

You are advised to have read Section 2 of W as far as equation (2.13) on W12 and the associated notes in Unit 1 thoroughly before listening to this programme. Have W on hand during the broadcast as extensive reference is made to it. The figures below may also be useful.

Ralph Smith introduces the course, laying emphasis on the fact that the student is not expected to attempt every self-assessment question in each unit. Geoff Moss then discusses d'Alembert's solution of the wave equation as applied to a finite string and describes the behaviour expected of the string when it is initially at rest and has the shape of a pulse function (a pulse function about some point x_1 is a function which has image zero except in some small neighbourhood of x_1). Ralph Smith reviews the means by which d'Alembert's solution can be extended to take account of fixed boundary conditions. This procedure is applied by Geoff Moss to explain how the moving wave pulses generated by the original pulse function bounce off the fixed ends of the string. The subsequent motion of the pulses is then outlined. The programme is concluded by Ralph Smith, who invites students to comment on the course's broadcasts.

During the programme reference is made to equations (2.1), (2.5), (2.6), (2.7), (2.8), (2.9), (2.11) and (2.13) of *W*.

The moving points referred to in the programme are shown below:

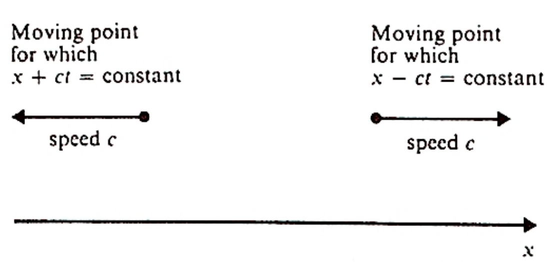


Fig. 1

If $f(x)$ is a pulse function of height H about the point $x = x_1$ it will be of the form shown in Fig. 2(a); the graph of $u(x, t)$ against x for fixed $t > 0$ in d'Alembert's solution with f a pulse function and $g = 0$ will be as in Fig. 2(b).

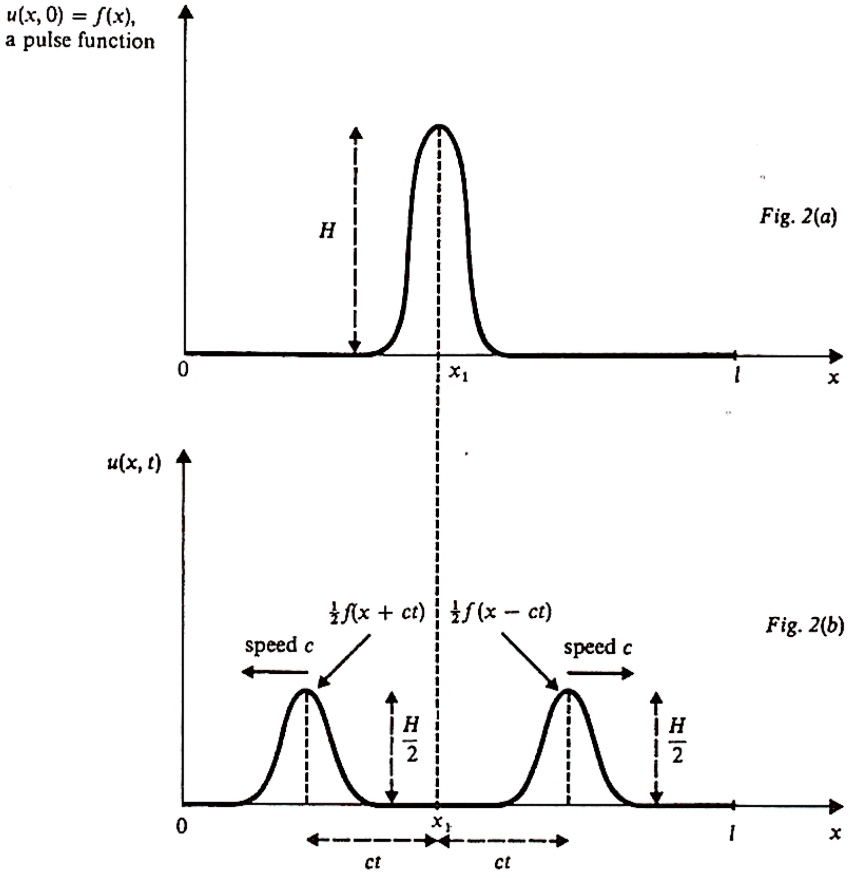


Fig. 2(a)

Fig. 2(b)

A pulse is approaching the end $x = 0$ of the string where a fixed boundary condition holds ($u(0, t) = 0$ for $t \geqslant 0$). Just before the leftward bound pulse hits the boundary the situation is as in Fig. 3(a). Just afterwards a rightward bound pulse of the opposite sign has emerged. This is shown in Fig. 3(b).

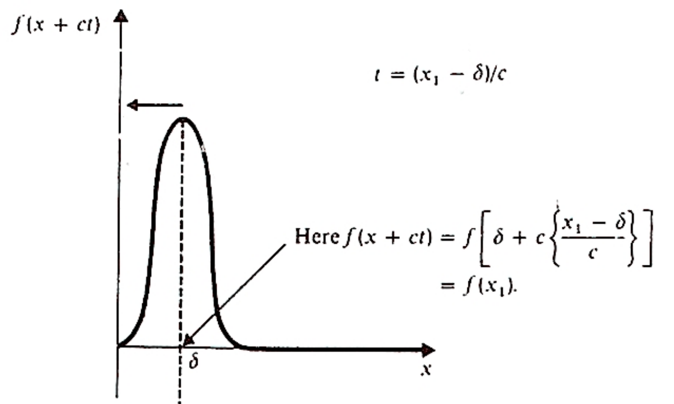


Fig. 3(a)

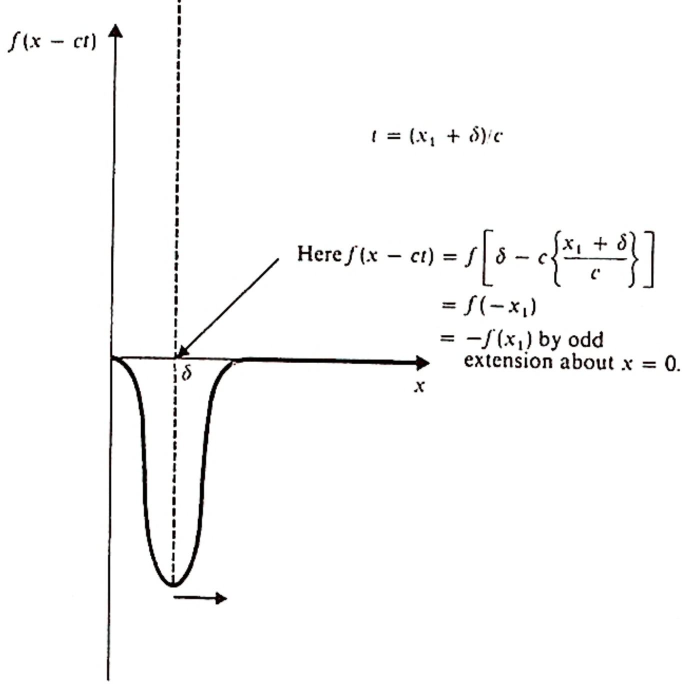


Fig. 3(b)

M321 R2

RADIO PROGRAMME 2

Broadcast Notes

LINE AND SURFACE INTEGRALS

Ralph Smith introduces the concept of a line integral and after giving a definition looks at a specific example. Peter Smith gives the mathematical description of a surface S, describes how an element of area ΔS can be evaluated and works through a surface integral problem. Reference is made to the following notes in the programme.

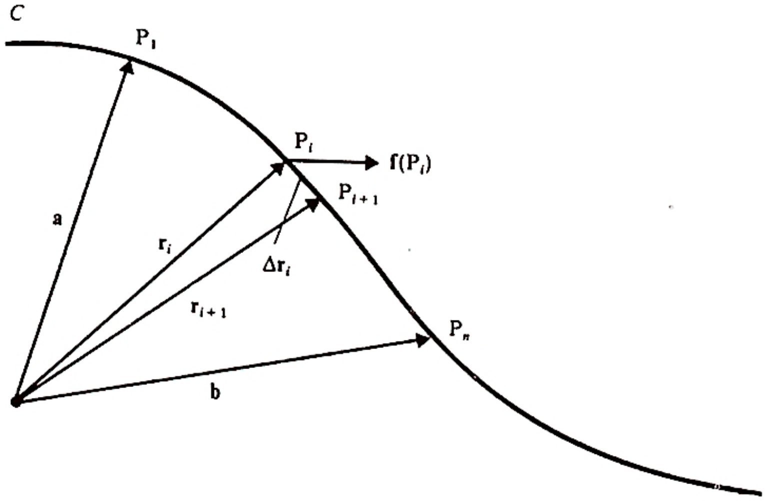


Fig. 1

Work done along $\overrightarrow{P_iP_{i+1}} \simeq \mathbf{f}(P_i) \cdot \Delta\mathbf{r}_i$

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \lim_{\|\Delta \mathbf{r}_i\| \to 0} \sum_i \mathbf{f}(P_i) \cdot \Delta\mathbf{r}_i \qquad (1)$$

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_a^b \mathbf{f}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt \qquad (2)$$

Example

Evaluate $\int_C \mathbf{f} \cdot d\mathbf{r}$ when $\mathbf{f}(x, y, z) = (y, x, 1)$ and C is given by

$$\mathbf{r}(t) = (\cos t, \sin t, t) \quad \text{for } 0 \leqslant t \leqslant 4\pi. \qquad (3)$$

Solution

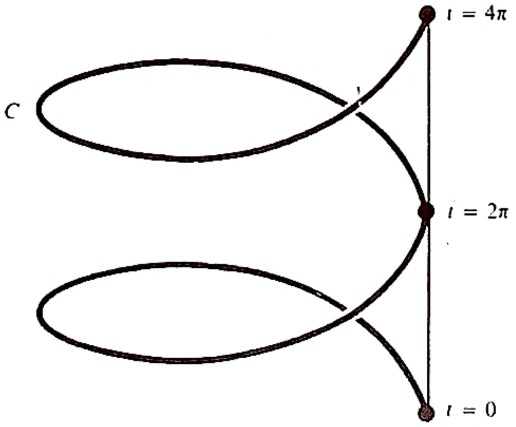


Fig. 2

$$\mathbf{f}(\mathbf{r}(t)) = (\sin t, \cos t, 1)$$

$$\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t, 1) \tag{4}$$

$$\mathbf{f} \cdot \frac{d\mathbf{r}}{dt} = -\sin^2 t + \cos^2 t + 1 = \cos 2t + 1 \tag{5}$$

$$\int_0^{4\pi} (\cos 2t + 1)\, dt = \left[\frac{\sin 2t}{2} + t\right]_0^{4\pi} = 4\pi \tag{6}$$

Exercise

Evaluate $\int_C \mathbf{f} \cdot d\mathbf{r}$ when $\mathbf{f}(\mathbf{r}(t)) = (\sin t, \cos t, 1)$ and C is given by

$$\mathbf{r}(t) = (1, 0, t) \quad \text{for } 0 \leqslant t \leqslant 4\pi. \tag{7}$$

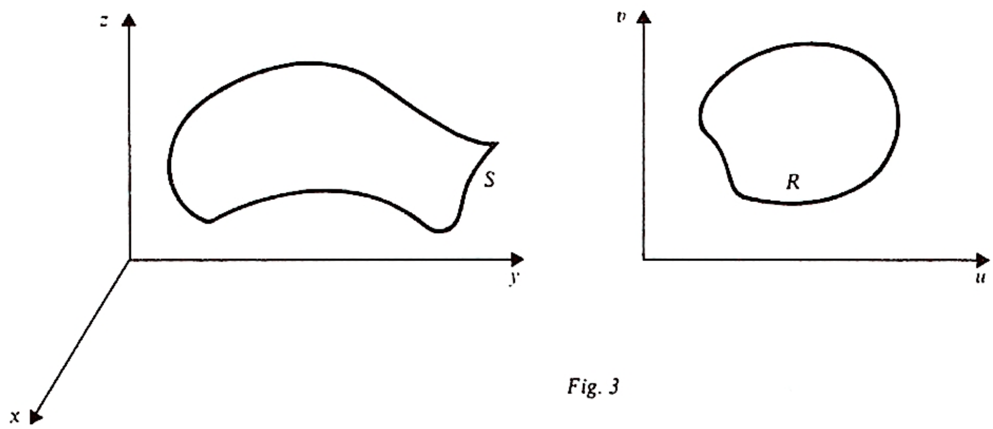


Fig. 3

$$\mathbf{r} = (a \cos u \sin v, a \sin u \sin v, a \cos v) \tag{8}$$

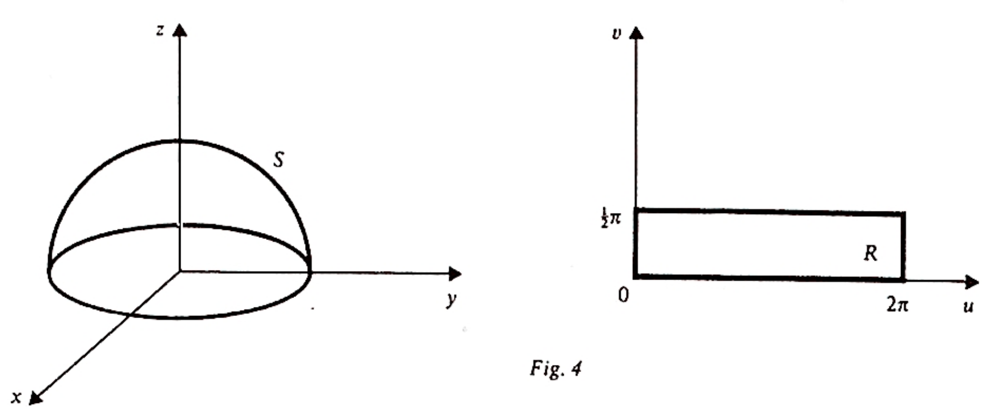


Fig. 4

The integral of a scalar field ϕ over a surface S is written as

$$\iint_S \phi \, dS.$$

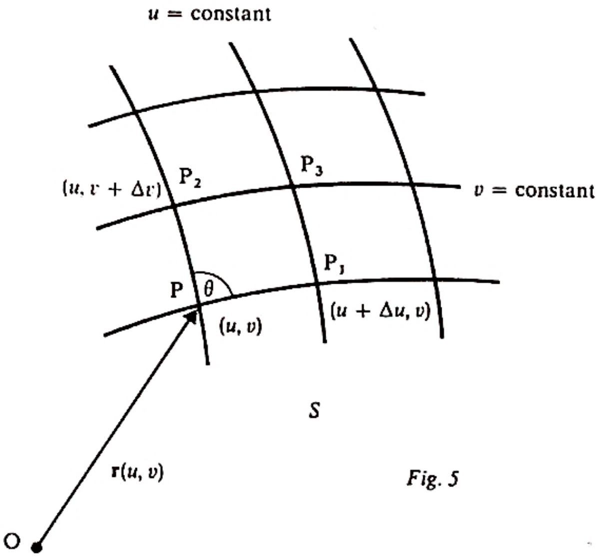


Fig. 5

$$\begin{aligned}\Delta S = \text{area of } P P_1 P_3 P_2 &\simeq |\overrightarrow{PP_1}|\,|\overrightarrow{PP_2}| \sin\theta \\ &= |\overrightarrow{PP_1} \times \overrightarrow{PP_2}| \\ &= |(\mathbf{r}_1 - \mathbf{r}) \times (\mathbf{r}_2 - \mathbf{r})| \\ &\simeq \left|\frac{\partial \mathbf{r}}{\partial u}\Delta u \times \frac{\partial \mathbf{r}}{\partial v}\Delta v\right| \\ &= \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \Delta u \Delta v.\end{aligned}$$

Total area of $S = \displaystyle\iint_R \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv.$

The integral of the scalar field ϕ over S is

$$\iint_S \phi \, dS = \iint_R \phi \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv$$

Example

Evaluate $\displaystyle\iint_S \phi \, dS$ where $\phi = x^2$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution

The surface can be represented parametrically as

$$\mathbf{r} = (a \cos u \sin v, a \sin u \sin v, a \cos v) \qquad 0 \leqslant u \leqslant 2\pi, 0 \leqslant v \leqslant \pi.$$

On the surface,

$$\phi = a^2 \cos^2 u \sin^2 v.$$

Now

$$\frac{\partial \mathbf{r}}{\partial u} = (-a \sin u \sin v, a \cos u \sin v, 0),$$

$$\frac{\partial \mathbf{r}}{\partial v} = (a \cos u \cos v, a \sin u \cos v, -a \sin v),$$

so

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u \sin v & a \cos u \sin v & 0 \\ a \cos u \cos v & a \sin u \cos v & -a \sin v \end{vmatrix}$$

$$= a^2[-\cos u \sin^2 v\mathbf{i} - \sin u \sin^2 v\mathbf{j} - \sin v \cos v\mathbf{k}],$$

and

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| = a^2(\sin^4 v + \sin^2 v \cos^2 v)^{\frac{1}{2}}$$

$$= a^2 \sin v.$$

We can now evaluate the integral:

$$\iint_S \phi \, dS = \iint_R \phi \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv$$

$$= \int_0^{\pi} \int_0^{2\pi} (a^2 \cos^2 u \sin^2 v)(a^2 \sin v)\, du\, dv$$

$$= a^4 \int_0^{2\pi} \cos^2 u\, du \int_0^{\pi} \sin^3 v\, dv.$$

Now

$$\int_0^{2\pi} \cos^2 u\, du = \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2u)\, du = \pi,$$

$$\int_0^{\pi} \sin^3 v\, dv = -\int_{v=0}^{v=\pi} (1 - \cos^2 v) d(\cos v)$$

$$= -[\cos v - \tfrac{1}{3}\cos^3 v]_{v=0}^{v=\pi}$$

$$= 2 - \tfrac{2}{3} = \tfrac{4}{3},$$

so that

$$\iint_S \phi\, dS = \tfrac{4}{3}\pi a^4.$$

M321 TV1

TELEVISION PROGRAMME 1

Broadcast Notes

BOUNDARY CONDITIONS AND PROPERLY POSED PROBLEMS

TV1 may be considered revision of certain aspects of the first three units of the course. In the programme we examine

(a) the way in which boundary and initial conditions determine a unique solution of the partial differential equation modelling a physical situation,

(b) what happens when there are too few boundary conditions,

(c) what happens when there are too many boundary conditions,

(d) the importance of continuity of a solution with respect to the data from which it is derived.

The ideas are tackled through examples in each case. Ralph Smith introduces the programme and outlines its contents.

Determining a Unique Solution

Daniel Lunn discusses various physical phenomena with wave characteristics, all satisfying the wave equation

$$u_{tt} - c^2 u_{xx} = 0.$$

[A subscript notation for partial derivatives is used throughout the programme. Thus

$$u_t = \frac{\partial u}{\partial t},\ u_{xx} = \frac{\partial^2 u}{\partial x^2} \text{ etc.]}$$

He displays several solutions of this equation and explains that the correct solution corresponding to any given physical situation is deduced by using the associated boundary and initial conditions.

This is illustrated by the case of a string fixed at both ends. The appropriate boundary conditions are

$$u(0, t) = u(l, t) = 0.$$

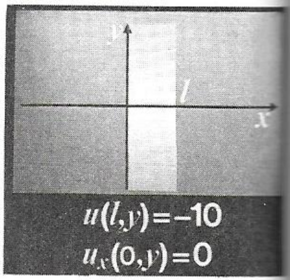
The string could be set in motion by being plucked (drawn aside from its equilibrium position and released from rest) or by receiving an initial impact while in its equilibrium state. The initial conditions corresponding to these two possibilities would have the forms

$$u(x, 0) = f(x) \qquad 0 \leqslant x \leqslant l,$$

$$\frac{\partial u}{\partial t}(x, 0) = u_t(x, 0) = g(x) \qquad 0 \leqslant x \leqslant l,$$

respectively. In general both types of initial condition are required. The four conditions imposed on the string are then sufficient to determine a unique solution to the wave equation.

Too Few Boundary Conditions



To show that this approach does not always work, Geoff Moss considers the physical situation of a long narrow ice rink of width l, insulated along one long side and cooled to a temperature of $-10°$C on the other by a pipe carrying a refrigerating substance. This can be modelled mathematically by an infinite strip in the xy-plane; we suppose that after any initial fluctuations the steady-state heat conduction equation for two dimensions will hold. This is just Laplace's equation

$$u_{xx} + u_{yy} = 0. \tag{1}$$

[This is not strictly speaking the correct equation, since unless the surface of the rink is insulated, or equivalently the atmosphere is at a constant temperature of $-10°$C, there will be a transfer of heat at the interface between the ice and the air above it. A suitable equation to describe this situation would be $u_{xx} + u_{yy} = h(u - u_0)$, where u_0 is the atmospheric temperature and h is a constant. This is difficult to solve however, so we imagine a simplified physical situation in which no heat transfer takes place at the surface of the rink.]

The boundary condition due to the refrigerating pipe at $x = l$ is

$$u(l, y) = -10. \tag{2}$$

The boundary at $x = 0$ is assumed insulated. There cannot consequently be any flow of heat across this boundary. Since heat flow is proportional to temperature gradient, the component of temperature gradient perpendicular to $x = 0$ must be zero. That is

$$u_x(0, y) = 0. \tag{3}$$

The equation (1) together with the boundary conditions (2) and (3) has the simple solution

$$u(x, y) = -10 \tag{4}$$

but we also find the further solutions

$$u(x, y) = -10 + e^{(-\pi y/2l)}\cos\left(\frac{\pi x}{2l}\right) \tag{5}$$

$$u(x, y) = -10 - 3e^{(-3\pi y/2l)}\cos\left(\frac{3\pi x}{2l}\right). \tag{6}$$

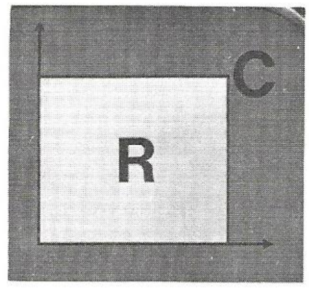
[N.B. There is an error in the TV display of equation (6), where a y appears in place of x on the right-hand side.]

There are many other solutions as well. In order to find a unique one we must impose further conditions. We see in equation (5) that $u \rightsquigarrow \infty$ as $y \rightsquigarrow -\infty$. In equation (6) u is in general unbounded as $y \rightsquigarrow -\infty$. [It is stated in the programme that $u \rightsquigarrow -\infty$ as $y \rightsquigarrow -\infty$ in equation (6). In fact this is only true for $0 \leqslant x < l/3$, and $u \rightsquigarrow \infty$ for $l/3 < x < l$. $u = -10$ for $x = l/3$.] In equation (4) however u is constant and hence bounded as $y \rightsquigarrow \pm\infty$. An infinite temperature has no physical meaning, so we impose the extra "boundary condition":

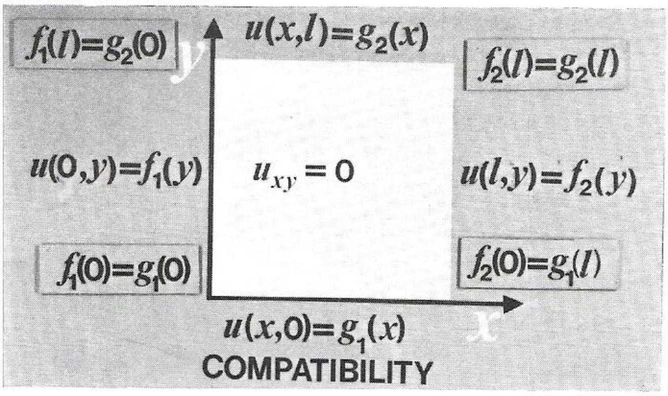
u remains finite as $y \rightsquigarrow \pm\infty$.

We find then $u(x, y) = -10$ to be the unique solution of the problem. The non-uniqueness arose from specifying too few boundary conditions.

Too Many Boundary Conditions



Daniel Lunn next investigates whether it is possible to have too many conditions. He tries to solve a Dirichlet problem for the partial differential equation $u_{xy} = 0$, i.e. he seeks a solution in the region R subject to its values being known at all points of the boundary C. The boundary values for u are chosen in the most general possible form, subject to the condition that u should be continuous along the boundary. R in this case is a square of side l. To make u continuous along C we require the functions f_1, f_2, g_1, g_2 to satisfy compatibility conditions (see picture).



The general solution of $u_{xy} = 0$ is

$$u(x, y) = p(x) + q(y),$$

where p and q are arbitrary functions to be determined by the boundary conditions. Putting $x = 0$ gives

$$p(0) + q(y) = f_1(y) \tag{7}$$

and putting $y = 0$,

$$p(x) + q(0) = g_1(x). \tag{8}$$

We can then find p and q from these equations, but there remains one arbitrary constant in these functions (if $p(x)$ and $q(y)$ satisfy the equations then $p(x) + K$ and $q(y) - K$ will also do so, K being any constant). Choosing $q(0) = 0$ we find from (8) that

$$p(x) = g_1(x)$$

and then from (7) that

$$q(y) = f_1(y) - g_1(0).$$

Hence we obtain the unique solution to the problem.

$$u(x, y) = g_1(x) + f_1(y) - g_1(0).$$

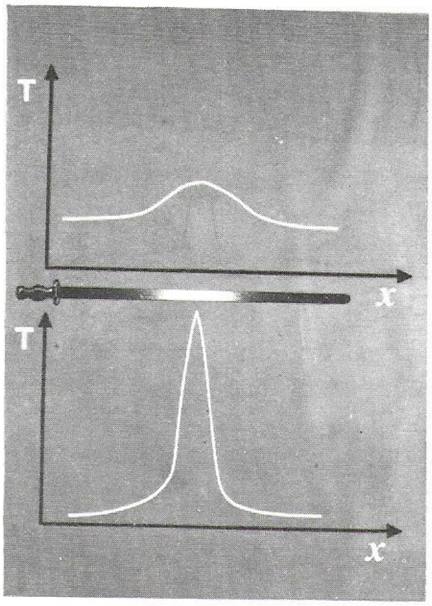
[Since $u(x, y) = p(x) + q(y)$, the arbitrary constant K mentioned above cancels out in the final solution. It is not therefore necessary to choose $q(0) = 0$; we could instead just add equations (7) and (8) and substitute for $p(0) + q(0)$ using (8) with $x = 0$.]

We have apparently obtained a unique solution using only two of the four boundary conditions. Since the remaining two conditions express u in terms of arbitrary functions, the "solution" will fail in general to satisfy them. The problem is overdetermined and it is not in fact meaningful to talk of a solution. Dirichlet's problem for the equation $u_{xy} = 0$ has no solution.

Continuity With Respect to the Data

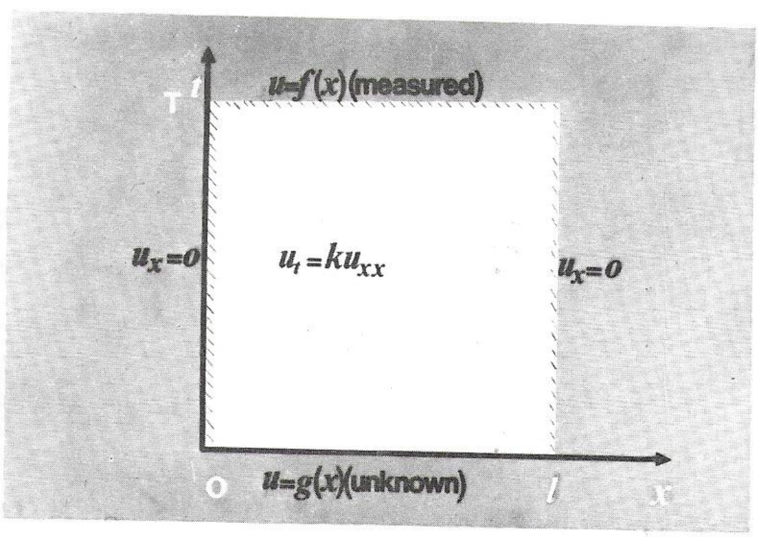
So far the need to investigate both the existence and uniqueness of solutions has been demonstrated. It remains to show that a useful solution should be "continuous with respect to the data". Dominic Jordan defines this last phrase as meaning that if the initial or boundary conditions change by small amounts, then the corresponding solution experiences changes of the same order of magnitude. He describes it more loosely as "small causes produce small effects" and refers to Weinberger for a precise statement.

The principle is illustrated by looking at a method for evaluating the temperature distribution in a fire. A poker is pushed into the fire and left to settle down. When it is pulled out some time later it will contain an approximate record of temperature

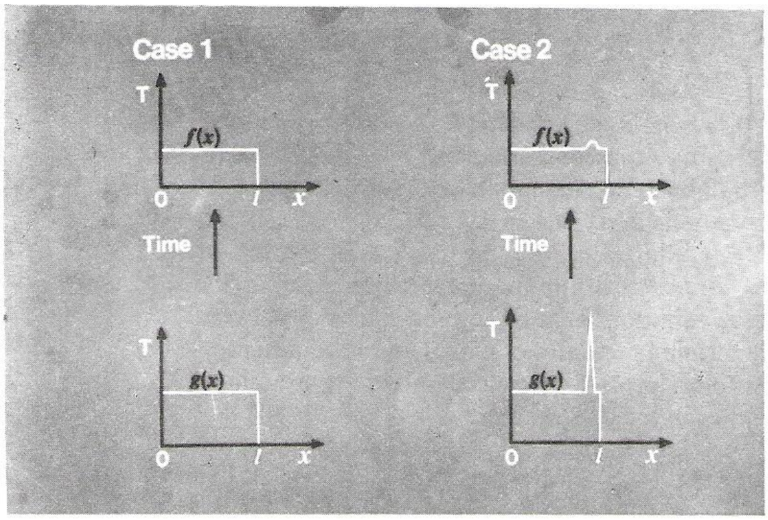


distribution at the instant of withdrawal. This record will rapidly deteriorate; heat loss could theoretically be prevented by suitable insulation, but heat would still transfer from hotter to colder regions. Even so an uneven distribution would remain, and it might be possible to deduce the state of the fire from it. The first thing to do is to find out whether the problem is well posed.

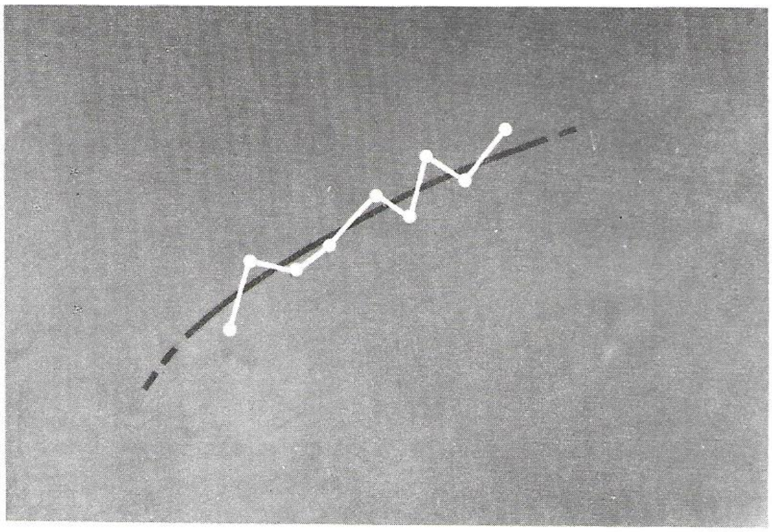
At $t = 0$ the temperature $u(x, 0)$ is given by a function $g(x)$. We suppose this to be the unknown temperature of the fire. At a later time the distribution is given by $f(x)$, which is measurable. In the meantime we assume the poker to be completely insulated, so that $u_x = 0$ at the ends and the heat conduction equation is obeyed. We seek $g(x)$



given $f(x)$ and it looks like an initial-boundary value problem turned upside down. It soon becomes apparent that small changes in f will signify large changes in g; one way of showing this is to imagine two different fires, whose pokers show even temperature distributions save for a small narrow bump in one case. We infer the temperature distributions in the fires to be fairly even, except for a hot spot with a



very high temperature in the fire whose poker displayed the small irregularity. It seems that the problem is not well posed, and that the solution will not be continuous with respect to the data. This is significant even in the absence of hot spots in the fire, since the data function $f(x)$ would in practice consist of measurements subject to small random errors. The computed data would be only an approximation



to the actual temperature distribution at the time of measurement, and a computer programmed with a scheme to obtain $g(x)$ from $f(x)$ would determine a closely spaced series of spurious peaks.

Dominic Jordan concludes by inviting you to turn the "upside-down initial value problem" the right way up by using the variable $-t$ in place of t, comparing your findings with statements on *W61*.

Geoff Moss and Daniel Lunn summarise the programme. We have emphasised the role of boundary conditions in partial differential equations and the need to know that a problem is well posed. Only if a solution exists uniquely and is continuous with respect to the data do we have a properly posed problem, and only then can we model a given physical situation with any validity mathematically.

M321 R3

RADIO PROGRAMME 3

Broadcast Notes

FINITE-DIFFERENCE METHODS

The programme is linked with *Unit 5*, the first numerical unit of the course. After an introduction by Peter Thomas, Leslie Fox proceeds to demonstrate that the finite-difference methods used to solve partial differential equations are an extension of the methods for ordinary differential equations described in *Unit M201 21, Numerical Solution of Differential Equations.* He illustrates this by discretizing the one-dimensional heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$ first in the x-direction, then in the t-direction. Peter Thomas discusses the variations in accuracy and efficiency between different finite-difference schemes, describing measures of global and local accuracy Finally the circumstances in which local errors fail to accumulate are examined. Reference is made to the following notes in the programme.

The ordinary differential equation

$$y'(x) + f(x)y(x) = g(x) \tag{1}$$

together with the initial condition $y(x_0) = \alpha$ gives a unique solution. The first order Taylor approximation to the true solution is

$$y(x_1) \equiv y(x_0 + k) \simeq y(x_0) + ky'(x_0) \tag{2}$$

where $x_1 = x_0 + k$.

Substitute this into equation (1) to obtain

$$y_1 \equiv y(x_1) \equiv y(x_0 + k) \simeq y(x_0) + k\{g(x_0) - f(x_0)y(x_0)\}. \tag{3}$$

Now expand in a Taylor series about x_1, repeat the substitution, and so on:

$$\begin{array}{cccccc} y_2 & \equiv y(x_2) & \equiv y(x_1 + k) & \simeq y(x_1) & + k\{g(x_1) & - f(x_1)y(x_1)\} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{r+1} & \equiv y(x_{r+1}) & \equiv y(x_r + k) & \simeq y(x_r) & + k\{g(x_r) & - f(x_r)y(x_r)\} \end{array} \tag{4}$$

where $x_r = x_0 + rk$, $r = 1, 2, \ldots$.

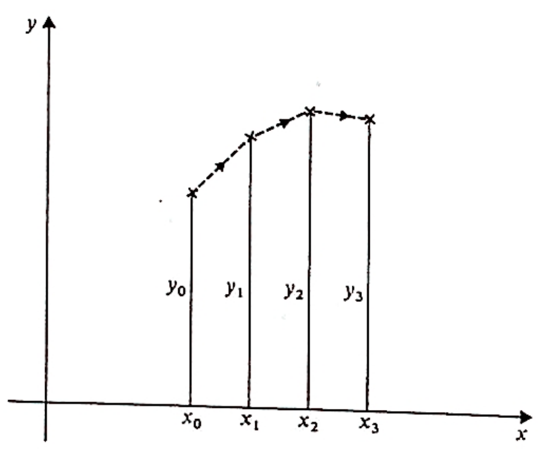


Fig. 1

Curve fitting to the points $\{(x_r, y_r): r = 0, 1, 2, \ldots\}$ gives an approximation to the true solution.

Turning now to partial differential equations, consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{5}$$

We wish to find values of $u(x, t)$ in the region shown in Fig. 2, with specified initial and boundary values.

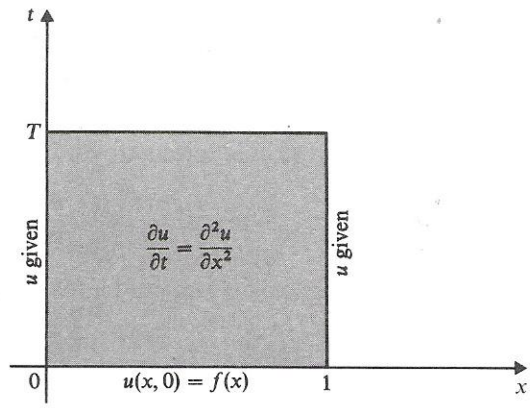


Fig. 2

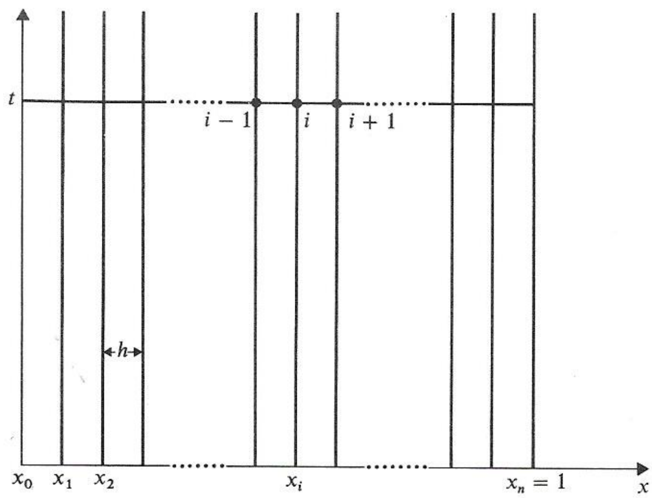


Fig. 3

Equation (5) is to be reduced to a system of ordinary differential equations. Discretizing in the x-direction gives $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$. With

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{6}$$

equation (5) becomes

$$\frac{\partial u_i}{\partial t} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{7}$$

This is an ordinary differential equation, because u_i is a function of t only. It can therefore be written

$$\frac{du_i}{dt} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{8}$$

There is one of these equations for each value of i from 1 to $n - 1$. The system can be written

$$\frac{du_i}{dt} + \frac{2}{h^2} u_i = \frac{u_{i+1} + u_{i-1}}{h^2} \qquad i = 1, 2, \ldots, n - 1. \tag{9}$$

u_0 and u_n are specified for all $t > 0$ and u_i is a function of t along the line $x = x_i$.

We can write this system of equations in matrix form as

$$\frac{d\mathbf{u}(t)}{dt} = A\mathbf{u}(t) + \mathbf{b}(t) \qquad i = 1, 2, \ldots, n - 1 \tag{10}$$

where A is the tridiagonal matrix

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix},$$

$\mathbf{u}(t)$ is the vector $\begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1}(t) \end{bmatrix}$ and $\mathbf{b}(t)$ the vector $\frac{1}{h^2}\begin{bmatrix} u_0(t) \\ 0 \\ \cdot \\ \cdot \\ 0 \\ u_n(t) \end{bmatrix}$.

We next use the previous discussion of ordinary differential equations to discretize in the t-direction. [u consequently acquires a second suffix. At both stages of the discretization a suffix ranging over a discrete set of values replaces a variable which may take any value in a closed interval. Thus when we discretized in the x-direction $u(x, t)$ became $u_i(t)$. This now becomes $u_{i,r}$ as we discretize in the t-direction.]

Applying equation (4) to equation (9) gives

$$u_{i,r+1} = u_{i,r} + k\left\{\frac{u_{i+1,r} + u_{i-1,r}}{h^2} - \frac{2}{h^2}u_{i,r}\right\}$$

or

$$u_{i,r+1} = u_{i,r} + \frac{k}{h^2}\{u_{i+1,r} - 2u_{i,r} + u_{i-1,r}\} \qquad (11)$$

where the suffices can be identified using Fig. 4.

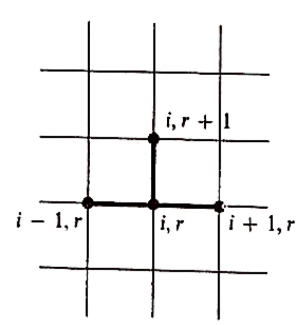


Fig. 4

Problem

Show that use of the Trapezoidal Rule method given by

$$y_{r+1} = y_r + \tfrac{1}{2}k(y'_r + y'_{r+1})$$

to solve equation (9) produces the Crank-Nicolson implicit finite-difference formula.

The global error $e_{i,j}$ of a finite-difference scheme is defined as

$$e_{i,j} = U_{i,j} - u_{i,j} \tag{12}$$

where U is the true solution of the partial differential equation concerned and u is the solution obtained from using the finite-difference scheme. The local error of a finite-difference scheme at a point is the addition to the global error caused by an application of the approximating formula at the point. This is illustrated by returning to the case of the one-dimensional heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \tag{13}$$

approximated by the simple explicit scheme

$$u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}). \tag{14}$$

where $r = \dfrac{k}{h^2}$.

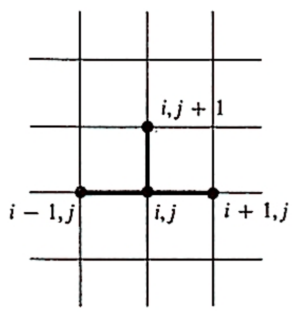


Fig. 5

Assume that we know the true solution U at all points along $t = jk$ and substitute the values into equation (14) to give

$$u^*_{i,j+1} = U_{i,j} + r(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}). \tag{15}$$

The error in $u^*_{i,j+1}$ as an approximation to $U_{i,j+1}$ is

$$U_{i,j+1} - u^*_{i,j+1} = U_{i,j+1} - U_{i,j} - r(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}). \tag{16}$$

By comparing equations (14) and (16) we see that the local error made in using the explicit scheme once is obtained by substituting the true solution of the partial differential equation into the finite-difference scheme. This quantity is one form of the local truncation error, $T^*_{i,j+1}$:

$$T^*_{i,j+1} = U_{i,j+1} - U_{i,j} - r(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}) \tag{17}$$

for the simple explicit scheme (14). We may also obtain a different form of local truncation error, $T_{i,j}$, by using the finite-difference equation

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \tag{18}$$

and defining

$$T_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{k} - \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}. \tag{19}$$

We see that

$$T^*_{i,j+1} = kT_{i,j} \tag{20}$$

for the schemes (14) and (18).

Problem

What would be the equivalent relationship to equation (20) for the simple explicit scheme

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

applied to the hyperbolic equation

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}?$$

M321 TV2

TELEVISION PROGRAMME 2

Broadcast Notes

CASE STUDY

This programme and Radio Programme 5 which follows it examine pressure transmission in an apparatus used for assessing air flow in mines. The physical situation is modelled mathematically and a nonlinear partial differential equation results. An analytic solution is found to the linearized equation (since the nonlinear term is small it may be ignored to a first approximation). Numerical methods are then used to solve both the linearized problem and the original equation. The results are finally compared with data obtained from an experiment. Ralph Smith introduces the programme.

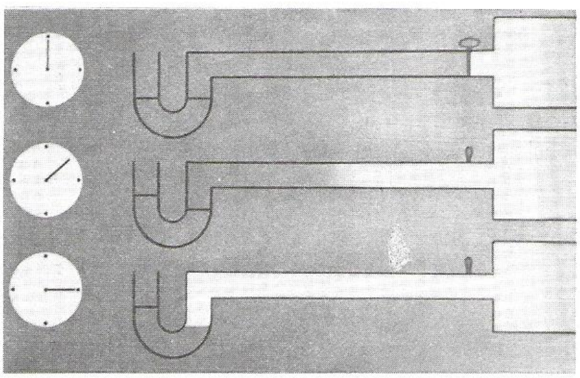
The Physical System

We move to a simulation tunnel in the British Coal Board's research institute at Swadlingcote (not "down a coalmine" as is suggested). Peter Thomas explains the main features of the apparatus. This consists of a long length of thin plastic tubing (initially out of focus in the right-hand half of the screen) with a manometer (which measures pressure differences) connected to one end. A tap at the other end of the tube is initially closed. After it has been opened the manometer will eventually register the pressure difference in the tunnel between the two ends of the tube, but there is a time lag before the levels of the liquid in the manometer re-approach a stationary position.

We are interested in measuring the *response time* of this apparatus, i.e. the time which it takes the manometer to register some given percentage of the total pressure difference (90 per cent is taken in the programme).

Mathematical Modelling

Back in the studio Peter Smith begins the mathematical modelling process with a representation of the apparatus. In the first instance one might expect the wave equation to model adequately the motion of the gas down the tube, the assumption being that the gas travels at the speed of sound, 340 m s^{-1}. Such a hypothesis gives results which bear no relation to experimental evidence however; the response times are much larger than predicted. The only likely explanation is that viscosity plays a part due to the small size of the tube.



Three equations are used to describe the motion of the gas. The first of these gives the total flux or flow rate q in terms of the coefficient of viscosity of the gas η, the radius of the tube a and the pressure gradient p_x (as in Television Programme 1 a subscript

notation is used to denote partial derivatives). This equation, which was derived for the case of steady flow in *Unit 3*, is

$$q = -\frac{\pi a^4}{8\eta} p_x. \tag{1}$$

[It is shown in the solution to SAQ 14 on page 39 of *Unit 3* that the flux Q through a circular pipe of radius a is given by $\frac{1}{8}k\pi a^4$; k here is a positive constant equal to $p/\rho\nu$ where p is the pressure gradient, ρ the density of the fluid and ν its kinematic viscosity (see page 20, lines -6, -5 of *Unit 3*). In the present restatement of the equation Q has been replaced by q, p describes the gas pressure (instead of the pressure gradient) so that the pressure gradient has magnitude $|p_x|$, and the kinematic viscosity ν is replaced by the (constant) coefficient of viscosity η, where $\eta = \rho\nu$. The minus sign is necessitated by the fact that there will be a flow of fluid in the positive x-direction only if the pressure decreases with x.]

$$\rho_t = \frac{-1}{\pi a^2}(\rho q)_x \qquad \frac{p}{\rho} = \frac{p_0}{\rho_0}\text{(Constant)}$$

$$p_t = D(pp_x)_x, D = \frac{a^2}{8\eta} \qquad q = -\frac{\pi a^4}{8\eta} p_x$$

The second equation relates the density ρ and the flow rate q:

$$\rho_t = -\frac{1}{\pi a^2}(\rho q)_x. \tag{2}$$

[This is the equation of conservation of mass and not the equation of motion as is erroneously stated in the programme; equation (1) is the equation of motion.]

Equation (2) is derived by considering a small section of the tube of length δx. The mass of this section of gas is $\pi a^2 \delta x \rho(x, t)$ so that the rate of increase of mass with respect to time is $\pi a^2 \rho_t \delta x$. We now obtain a second expression for this rate of increase. Mass flows into the left-hand end of the small section at a rate $\rho(x, t)q(x, t)$ (remember that q is the integral of fluid velocity over the cross-sectional area of the pipe and consequently expresses the *volume* flow rate) and flows out the right-hand end at a rate

$$\rho(x + \delta x, t)q(x + \delta x, t) = \rho(x, t)q(x, t) + \delta(\rho(x, t)q(x, t)).$$

The net rate of influx of mass is therefore

$$\rho q - (\rho q + \delta(\rho q)) = -\delta(\rho q),$$

and we may use Taylor's theorem to approximate this by $-(\rho q)_x \delta x$. Equating the two expressions obtained for the rate of increase of mass and cancelling δx gives the required result.

Thirdly we assume that the temperature of the gas remains constant throughout the motion. Boyle's Law then states that the pressure of the gas is proportional to its density. We therefore have

$$\frac{p}{\rho} = \frac{p_0}{\rho_0} \tag{3}$$

where p_0, ρ_0 are the initial values of the pressure and density respectively (they are also the values of these quantities in the vicinity of the manometer (but outside the tube) for all times $t \geqslant 0$).

Elimination of ρ and q between equations (1), (2) and (3) gives a nonlinear partial differential equation with p as its dependent variable,

$$p_t = D(pp_x)_x \tag{4}$$

where $D = a^2/8\eta$ is a constant (not to be confused with any form of differential operator).

The transformation

$$P = \frac{p - p_0}{\Delta p_0} \tag{5}$$

where Δp_0 is the pressure difference in the tunnel between the two ends of the tube, is now used to simplify the problem. ε is written for the ratio $\Delta p_0/p_0$. Equation (4) becomes

$$P_t = \kappa[(1 + \varepsilon P)P_x]_x \tag{6}$$

where $\kappa = a^2 p_0/8\eta$ is a constant.

Since $p = p_0$ at $t = 0$ we see from (5) that

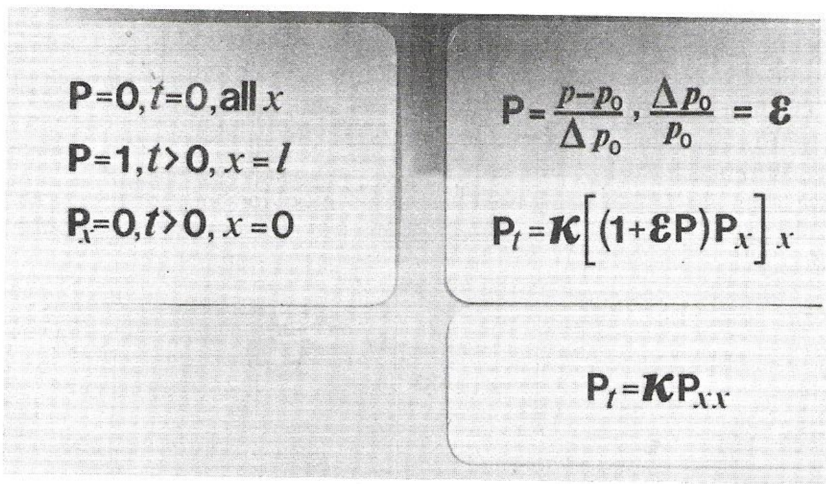
$$P(x, 0) = 0 \qquad 0 \leqslant x \leqslant l \tag{7}$$

where l is the length of the tube. After the tap is opened at $x = l$ the pressure there is maintained at $p_0 + \Delta p_0$ so that

$$P(l, t) = 1 \qquad t > 0. \tag{8}$$

At the manometer we assume that the flow of gas is zero, i.e. $q(0, t) = 0$. From equation (1) this means that $p_x(0, t) = 0$, and so from (5)

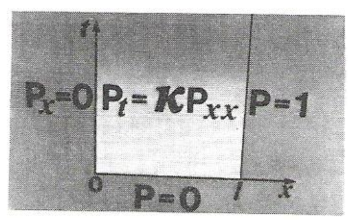
$$P_x(0, t) = 0 \qquad t > 0. \tag{9}$$



Equations (6) to (9) give a problem composed of a nonlinear partial differential equation together with initial and boundary conditions. In practice ε is found to be small so that it may be neglected to a first approximation. Equation (6) is then replaced by

$$P_t = \kappa P_{xx} \tag{10}$$

which is the heat conduction equation.



Equations (7) to (10) give a standard boundary value problem. The viewer is referred to the broadcast notes for its solution. [In fact the solution to this problem is derived in Radio Programme 5 and can be found in the corresponding broadcast notes.]

What Does the Model Predict?

At the manometer we find

$$P(0,t) = 1 - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{2}}\exp\left[-(n+\tfrac{1}{2})^2\pi^2\frac{\kappa t}{l^2}\right]. \tag{11}$$

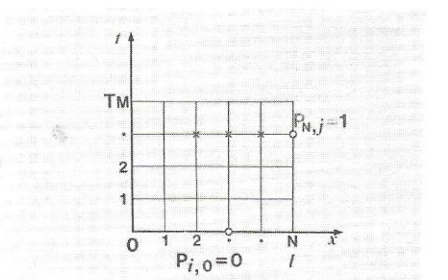
$P(0, t)$ approaches the value 1 as t becomes large. Suppose $t = t_R$ when $P(0, t)$ reaches some fixed percentage of its final limiting value, say 90 per cent for definiteness. Then $P(0, t_R) = 0.9$, and we can find a constant c such that

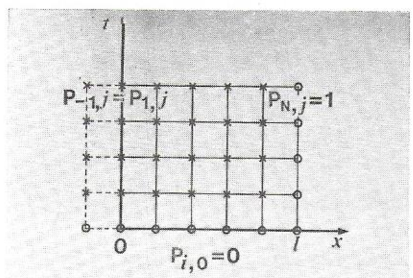
$$0.9 = 1 - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{2}}\exp[-(n+\tfrac{1}{2})^2\pi^2 c]. \tag{12}$$

It is clear that c is independent of the dimensions of the tube and of the response time. Putting $t = t_R$ in equation (11) and comparing with (12) we see that $\kappa t_R/l^2 = c$. Using the expression for κ below equation (6) it follows that the ratio $a^2 t_R/l^2$ is constant as the length l and the radius a are permitted to vary. In other words the response time t_R is proportional to the square of the length of the tube and inversely proportional to the square of its radius.

The Numerical Approach

Peter Thomas looks at the linearized problem given by equations (7) to (10) with a view to solving it numerically. A piece of computer-animated film shows the discretization of the rectangular region in which a solution is required and of the initial and boundary conditions. [Two points here: we cannot "take the response time to be T" as stated in the programme, since it is the response time which we wish to find by solving the problem. T should therefore be chosen to be considerably larger than any expected response time. The film successively assigns the values 1 and 0 to $P_{N,0}$. In practice $P_{N,0}$ is put equal to 0.5 rather than either of these values (see the notes to Library Program $RTT 321). We then have $P_{N,j} = 1$ only for $j = 1, 2, \ldots, M$, and not "from 0 up to M" as stated. P incidentally is being used here as a symbol for both the analytic and finite-difference solutions for the pressure difference.]





h and k are taken to be the (constant) mesh sizes in the x- and t-directions respectively, so that $Nh = l$, $Mk = T$. The derivative boundary condition at $x = 0$ is dealt with by using the central difference formula

$$(P_x)_{0,j} \simeq (P_{1,j} - P_{-1,j})/2h \tag{13}$$

and extending the mesh accordingly. The discretized initial and boundary conditions are

$$P_{i,0} = 0 \qquad i = -1, 0, 1, \ldots, N-1$$

$$P_{N,0} = 0.5$$

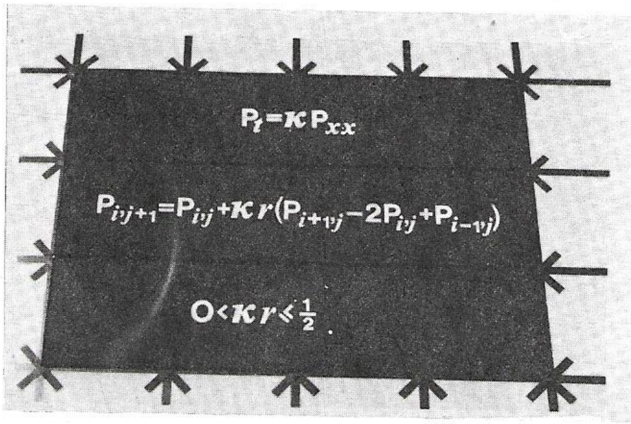
$$P_{N,j} = 1; \quad P_{-1,j} = P_{1,j} \quad j = 1, 2, \ldots, M. \tag{14}$$

Equation (10) is then replaced by the finite-difference scheme

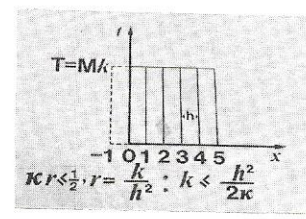
$$\frac{P_{i,j+1} - P_{i,j}}{k} = \frac{\kappa(P_{i+1,j} - 2P_{i,j} + P_{i-1,j})}{h^2}. \tag{15}$$

Putting the mesh ratio k/h^2 equal to r this can be rearranged as

$$P_{i,j+1} = P_{i,j} + \kappa r(P_{i+1,j} - 2P_{i,j} + P_{i-1,j}) \tag{16}$$



which is an explicit scheme. It is only stable when r satisfies the inequality $0 < \kappa r \leqslant \frac{1}{2}$, and this condition is somewhat prohibitive in terms of practical computing as is shown with a numerical example. The stability condition is $k \leqslant h^2/2\kappa$. Putting $\kappa = 100{,}000$ (a typical value), $l = 100$ m and $N = 5$, we have $h = l/5 = 20$ m, giving the restriction $k \leqslant 0.002$ s. Even if we take the equality here a value of $M = 1{,}250$ is needed in order to reach $T = 2.5$ s. Though this is a large value for M the situation becomes progressively worse if we seek a more accurate scheme.



If ten times as many grid points are taken on each time row we require M to be 125,000 (not "$1\frac{1}{4}$ million" as stated in the programme, but still too much for a computer to handle).

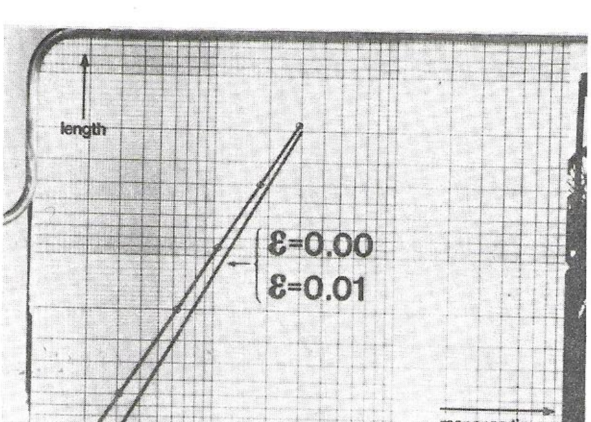
This stability condition is avoided by using an implicit scheme to replace equation (10) (for example the Crank–Nicolson scheme

$$\begin{aligned} -\kappa r P_{i-1,j+1} + (2 + 2\kappa r)P_{i,j+1} - \kappa r P_{i+1,j+1} &= \kappa r P_{i-1,j} \\ &\quad + (2 - 2\kappa r)P_{i,j} + \kappa r P_{i+1,j} \end{aligned} \tag{17}$$

is a satisfactory replacement, being stable for all values of r). In the nonlinear case the situation is very similar. A typical value of 0.01 is taken for ε. The initial and boundary conditions remain the same. An implicit scheme is again employed to avoid stability restrictions on the mesh size (although the viewer is referred to the broadcast notes for details of the implicit method in the nonlinear case this again is dealt with in Radio Programme 5).

Comparison With Experiment

We return to Swadlingcote to conduct an experimental test of our results. The pressure difference, created by a large fan, is known in advance and the manometer is marked at a point representing 90 per cent of the final reading. The time which it takes the liquid in the manometer to reach this level after the tap has been opened is recorded on a stopwatch.



Finally a set of such data for various different lengths of tube is compared with values obtained from the analytic and numerical solutions. The line on the left of the picture joins the experimental results while that on the right represents all three methods used in the programme; it is impossible to differentiate between the methods on this scale. [You may well wonder why these graphs are straight lines when it was shown earlier that at least in the analytic case the response time is proportional to the square of the length. The explanation is that the scales are not linear but logarithmic (this is the significance of the unequal mesh spacing on the graph paper). We can alternatively consider the graph as plotting $\log t_R$ against $\log l$ on linear scales. Since $t_R \propto l^2$, $\log t_R = 2 \log l + \text{constant}$, giving a straight line.] Since each method examined produced a result close to that of the experiment it is concluded that the linearization of the problem was a reasonable approximation to make. The viewer is referred to Radio Programme 5 for further discussion of the case study.

M321 R5

RADIO PROGRAMME 5

Broadcast Notes

CONTINUING THE CASE STUDY

This programme follows on directly from Television Programme 2. After an introduction by Peter Thomas, the physical aspects of the case study and its mathematical modelling are described again by Peter Smith, who subsequently solves the resulting (linearized) initial-boundary value problem using separation of variables. Peter Thomas then discusses the numerical solution of the nonlinear case, introducing a suitable implicit finite-difference scheme and describing how problems which arise in attempting to implement the scheme as a computer program can be overcome.

The paper by Jones and Jordan which is referred to in the programme is reproduced on the pages following these notes. This is followed by a listing of the library program $RTT 321, notes as to its use and a set of exercises. Reference is made to the following notes in the programme.

Figure 1 is a diagramatic representation of the apparatus.

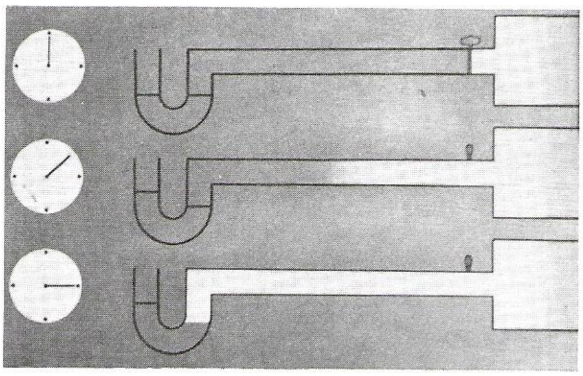


Fig. 1

The volume flow rate q in the pipe is given by

$$q = -\frac{\pi a^4}{8\eta}\frac{\partial p}{\partial x}, \tag{1}$$

where a is the radius of the tube, η the coefficient of viscosity and $\partial p/\partial x$ the pressure gradient.

We assume that the gas satisfies two further equations,

$$\frac{\partial \rho}{\partial t} = -\frac{1}{\pi a^2}\frac{\partial(\rho q)}{\partial x}, \tag{2}$$

and

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}, \tag{3}$$

which are deduced from the law of conservation of mass and Boyle's Law respectively (see the TV2 broadcast notes for more details about this and the following). The initial and boundary conditions for the pressure are

$$\begin{aligned} p(x, 0) &= p_0 && 0 \leqslant x \leqslant l, \\ p(l, t) &= p_0 + \Delta p_0 && t > 0, \\ \frac{\partial p}{\partial x}(0, t) &= 0 && t > 0. \end{aligned}$$

After eliminating ρ and q between equations (1), (2) and (3), and replacing the pressure p by the pressure difference $P = (p - p_0)/\Delta p_0$, we arrive at the nonlinear partial differential equation

$$\frac{\partial P}{\partial t} = \kappa \frac{\partial}{\partial x}\left[(1 + \varepsilon P)\frac{\partial P}{\partial x}\right] \tag{4}$$

where $\varepsilon = \Delta p_0/p_0$ and $\kappa = a^2 p_0/8\eta$. Neglecting the nonlinear term,

$$\frac{\partial P}{\partial t} = \kappa \frac{\partial^2 p}{\partial x^2} \tag{5}$$

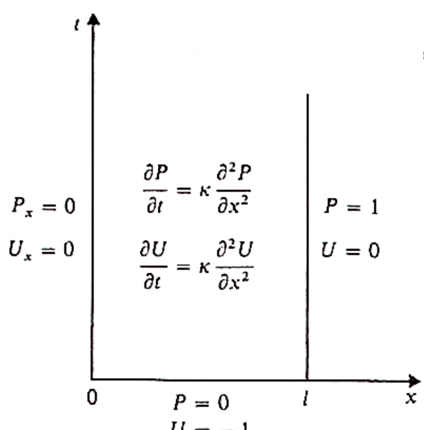


Fig. 2

For convenience P is replaced by $U = P - 1$, so that

$$U(l, t) = 0 \qquad t > 0,$$

$$\frac{\partial U}{\partial x}(0, t) = 0 \qquad t > 0,$$

$$U(x, 0) = -1 \qquad 0 \leqslant x \leqslant l.$$

Put

$$U(x, t) = F(x)G(t), \tag{6}$$

so that F and G satisfy

$$\frac{F''}{F} = \frac{G'}{\kappa G} = -c^2 \tag{7}$$

for some constant c. F satisfies

$$\frac{d^2F}{dx^2} + c^2 F = 0 \tag{8}$$

and the general solution of this is

$$F(x) = A\cos cx + B\sin cx. \tag{9}$$

The boundary condition at $x = 0$ implies that $B = 0$, and the condition at $x = l$ implies that $\cos cl = 0$. c can therefore take any of the values $\pi/2l$, $3\pi/2l$, $5\pi/2l, \ldots,$ or in general $(2n + 1)\pi/2l$ for integers n. Substituting this into the equation for G,

$$\frac{dG}{dt} + \kappa \frac{(2n + 1)^2\pi^2}{4l^2} G = 0. \tag{10}$$

This has the general solution

$$G = D_n \exp\left[-\kappa \frac{(2n + 1)^2\pi^2}{4l^2} t\right] \tag{11}$$

where D_n is a constant.

Adding all the separable solutions together we get

$$U(x, t) = \sum_{n=0}^{\infty} D_n \exp\left(\frac{-\kappa(2n + 1)^2\pi^2 t}{4l^2}\right) \cos\frac{(2n + 1)\pi x}{2l}, \tag{12}$$

and the D_n are chosen so that the cosine series obtained by putting $t = 0$ in equation (12) is equal to -1 on the interval $[0, l]$. The required series is

$$-1 = -4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\pi} \cos \frac{(2n+1)\pi x}{2l}, \tag{13}$$

so that the full solution is seen to be

$$P(x, t) = 1 - 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\pi} \exp\left(\frac{-\kappa(2n+1)^2\pi^2 t}{4l^2}\right) \cos \frac{(2n+1)\pi x}{2l}. \tag{14}$$

We find the pressure difference at the manometer by putting $x = 0$ in this equation. Note that the time t always appears in a factor $\kappa t/l^2$, so that this factor must always be the same value (for different lengths and radii of pipe) when $P(0, t)$ achieves any given value. The response time is therefore proportional to the square of the length of the tube and inversely proportional to the square of its radius.

For equation (4) there exists an implicit numerical scheme which is both stable and accurate but gives rise to systems of linear algebraic equations, namely

$$\begin{aligned} P_{i,j+1} - P_{i,j-1} = \tfrac{2}{3}\kappa r[\alpha^+\{(P_{i+1,j+1} - P_{i,j+1}) + (P_{i+1,j} - P_{i,j}) \\ + (P_{i+1,j-1} - P_{i,j-1})\} - \alpha^-\{(P_{i,j+1} - P_{i-1,j+1}) \\ + (P_{i,j} - P_{i-1,j}) + (P_{i,j-1} - P_{i-1,j-1})\}] \end{aligned} \tag{15}$$

where

$$\alpha^+ = 1 + \tfrac{1}{2}\varepsilon(P_{i+1,j} + P_{i,j}),$$
$$\alpha^- = 1 + \tfrac{1}{2}\varepsilon(P_{i,j} + P_{i-1,j}),$$

and $r = k/h^2$ (P is being used here as a symbol for both the analytic and finite-difference solutions for the pressure difference).

The initial and boundary conditions for the problem in terms of P are

$$\begin{aligned} P(x, 0) &= 0 \qquad 0 \leqslant x \leqslant l, \\ P(l, t) &= 1 \qquad t > 0, \\ \frac{\partial P}{\partial x}(0, t) &= 0 \qquad t > 0. \end{aligned} \tag{16}$$

The discontinuity in the data at $(l, 0)$ may affect the numerical solution, but this can be minimized by choosing a very small time step initially. Since equation (15) represents a three-level scheme we need to know values along $j = 0$ and $j = 1$ before we can start using it. Values for $j = 1$ are calculated from the known values on $j = 0$ by using a simple two-level scheme. We can construct a suitable explicit scheme as follows.

Writing equation (4) in the form

$$\frac{\partial P}{\partial t} = \kappa\left[\varepsilon\left(\frac{\partial P}{\partial x}\right)^2 + (1 + \varepsilon P)\frac{\partial^2 P}{\partial x^2}\right] \tag{17}$$

and replacing the first and second space derivatives by central differences,

$$\left(\frac{\partial P}{\partial x}\right)_{i,j} \simeq \frac{P_{i+1,j} - P_{i-1,j}}{2h}, \tag{18}$$

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_{i,j} \simeq \frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{h^2}, \tag{19}$$

we arrive at the scheme

$$\begin{aligned} P_{i,j+1} = P_{i,j} + \frac{\kappa r}{4}[\varepsilon(P_{i+1,j} - P_{i-1,j})^2 \\ + 4(1 + \varepsilon P_{i,j})(P_{i+1,j} - 2P_{i,j} + P_{i-1,j})]. \end{aligned} \tag{20}$$

Since the object of using an implicit scheme is to avoid the restrictions on step size which accompany explicit schemes we use the explicit scheme several times in order to provide values along the first time level of the implicit scheme.

[In fact we use the explicit scheme to provide values along the first and second time levels of the implicit scheme so that the latter escapes the immediate impact of the discontinuity in the initial and boundary data at $(l, 0)$.]

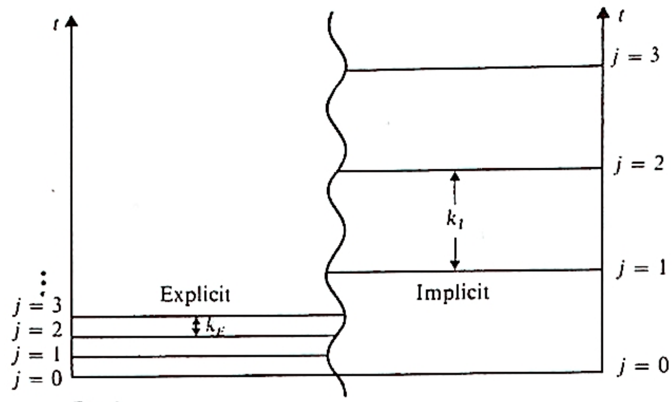


Fig. 3

The step length k_I of the implicit scheme must be an exact multiple of the step length k_E of the explicit scheme. The implicit scheme gives rise to a tridiagonal system of equations which can be solved by the recurrence method of *Unit 5*.

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Time lags in the transmission of pressure disturbances along long lengths of small bore tubing

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When it becomes necessary to transmit small air pressure differences over long distances the effect on the response time of the small bore tubing used to connect the source of pressure to a manometer should be known. This article develops a theoretical treatment for evaluating the time and shows it to vary directly as the square of the length and inversely as the square of the internal diameter of the tubing. The theory is confirmed as regards variation with length by laboratory experiments with Neoprene tubing having a bore of $\frac{3}{16}$ in. The experiments also include an examination of the effect of varying the magnitude of the pressure, the position of the manometer and the coiling of the tube.

1. Introduction

The transmission of differential air pressure to a distant position such as occurs in the remote control of fan speed or the recording at a distance of the pressure drop across an orifice plate are instances where long lengths of small bore tubing are used. The question arises as to how the tubing affects the response time of the manometer when lengths of tubing occasionally as great as one mile are used. Of special interest are the effects of the length and diameter of the tube on the time. This paper describes investigations, both theoretical and experimental, into this question for the case when small bore tubing is connected to a liquid-in-glass manometer whose volume change has been neglected.

2. Theoretical treatment for the propagation of pressure in a long small bore tube

It is assumed that the flow of air into the tube must, of course, be initiated by a wave motion, but as this is damped out relatively quickly compared to the times observed the characteristics of the air motion are not governed by the propagation of sound waves, shock waves or the like but by laminar viscous flow of air. It is shown later that the results obtained are consistent with this assumption.

The well-known Poiseuille expression for laminar flow is

$$q = -\frac{\pi a^4}{8\eta}\frac{\partial p}{\partial x} \tag{1}$$

where q is the volume flow rate in direction x, a the radius of the tube, η the dynamic viscosity of the air and p is the pressure, assumed in deriving this formula to be uniform over the cross section.

A second relation connecting pressure and flow rate is obtained as follows.

Consider a short portion of the tube between x and $x + \delta x$. The mass flow rate at x is ρq, where ρ is the density at x; and at $x + \delta x$ is $\rho q + \delta(\rho q)$. The net influx of mass into the volume $\pi a^2 \delta x$ per unit time is therefore $-\{\partial(\rho q)/\partial x\}\delta x$, and the rate of influx of mass per unit volume, which is equal to the rate of change of density, is $-(1/\pi a^2)\{\partial(\rho q)/\partial x\}$. Thus

$$\frac{\partial \rho}{\partial t} = -\frac{1}{\pi a^2}\frac{\partial(\rho q)}{\partial x}. \tag{2}$$

With isothermal conditions Boyle's law holds:

$$\frac{p}{\rho} = \text{const} = \frac{p_0}{\rho_0} \tag{3}$$

where p_0 and ρ_0 are reference values, say under the undisturbed conditions. Equation (2) can then be written in terms of pressure and flow rate:

$$\frac{\partial p}{\partial t} = -\frac{1}{\pi a^2}\frac{\partial(pq)}{\partial x} \tag{4}$$

and this is the second equation required.

Substitution into (4) from (1) gives

$$\frac{\partial p}{\partial t} = \left(\frac{a^2}{8\eta}\right)\frac{\partial}{\partial x}\left(p\frac{\partial p}{\partial x}\right) = D\frac{\partial}{\partial x}\left(p\frac{\partial p}{\partial x}\right) \tag{5}$$

where $D = a^2/8\eta$.

Consider that the end $x = l$ of the tube is subjected to sudden change of pressure from p_0 to $p_0 + \Delta p_0$. The other end, at $x = 0$, has a manometer attached of negligible volume so that the flow is always zero there, a condition which interpreted in terms of p by equation (1). Then the initial and boundary conditions to equation (5) are

$$\left.\begin{aligned} &p = p_0 \text{ at } t = 0, \quad \text{for all } x \\ &p = p_0 + \Delta p_0, \quad \text{for } t > 0, \quad \text{at } x = l \\ &\frac{\partial p}{\partial x} = 0, \quad \text{for } t > 0, \quad \text{at } x = 0 \end{aligned}\right\}. \tag{6}$$

Equation (5) is like the ordinary heat-conduction equation in a rod, with a variable diffusivity equal to Dp. The boundary conditions are analogous to a sudden change of temperature at one end, the other end being insulated. Equation (5) non-linear, but the non-linearity can be removed by demonstrating that its effect is negligible. Putting

$$P = \frac{(p - p_0)}{\Delta p_0}; \quad \frac{\Delta p_0}{p_0} = \varepsilon \tag{7}$$

then

$$\frac{\partial P}{\partial t} = \kappa\frac{\partial}{\partial x}\left\{(1 + \varepsilon P)\frac{\partial P}{\partial x}\right\} \tag{8}$$

with

$$\left.\begin{aligned} &P = 0 \text{ at } t = 0, \quad \text{for all } x \\ &P = 1 \text{ for } t > 0, \quad \text{at } x = l \\ &\frac{\partial P}{\partial x} = 0 \text{ for } t > 0, \quad \text{at } x = 0 \end{aligned}\right\} \tag{9}$$

where

$$\kappa = Dp_0 = \frac{a^2 p_0}{8\eta}. \tag{10}$$

Δp_0 will be of the order of a few inches of water gauge at most, so that ε, being the ratio of this quantity to atmospheric pressure, will be of order 1%, and it is intuitively clear that P will not exceed unity. The factor $1 + \varepsilon P$ in (8) is therefore effectively 1, and this makes the equation linear, its form being

$$\frac{\partial P}{\partial t} = \kappa \frac{\partial^2 P}{\partial x^2}. \tag{11}$$

The solution to this problem is given by Carslaw and Jaeger (1948, p. 87):

$$P(x, t) = \frac{2}{l} \sum_{n=0}^{\infty} (-1)^n \cos \beta_n x \{1 - \exp(-\kappa \beta_n^2 t)\}/\beta_n.$$

It can be verified that $\frac{2}{l} \sum_{n=0}^{\infty} (-1)^n (\cos \beta_n x)/\beta_n$ is identically equal to 1 so that

$$P(x, t) = 1 - \frac{2}{l} \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta_n} \exp(-\kappa \beta_n^2 t) \cos \beta_n x. \tag{12}$$

In the above $\beta_n = (2n + 1)\pi/2l$.

It is desired to find how long it takes for a change in pressure Δp_0 at $x = l$ to reach $x = 0$. At $x = 0$ the expression becomes

$$P(0, t) = 1 - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{1}{2})} \exp\{-(n + \tfrac{1}{2})^2 \pi^2 (\kappa t/l^2)\}. \tag{13}$$

The time taken for the pressure at $x = 0$ to attain a given fraction of its final value in cases where different tube diameters and lengths are considered is given by the relation

$$\kappa t/l^2 = \text{constant}$$

or

$$t \propto \frac{\text{length}^2}{\text{diameter}^2}. \tag{14}$$

Alternatively, dimensional arguments give this result directly.

Next, consider the actual times of propagation under circumstances similar to the tests described later.

For air, $\eta = 1 \cdot 3 \times 10^{-5}$ lb ft^{-1} sec^{-1}, $a = 1/128$ ft with the Neoprene tube available, and $p_0 = 6 \cdot 8 \times 10^4$ *pdl* ft^{-2}, so that $\kappa \simeq 4 \times 10^4$. Likely values of l lie between about 300 and 10000 ft, and the experiments that are described later in the paper give values of 90% response time between about 2 sec and 2×10^4 sec. Calling this time $T_{0 \cdot 9}$, the value of $\kappa T_{0 \cdot 9}/l^2$ has nearly the same value for all the experiments, as expected, the mean being about 0·90. For this value the first term of equation (13), corresponding to $n = 0$, has most significance, giving very nearly

$$P = \frac{p - p_0}{\Delta p_0} \simeq 1 - \frac{4}{\pi} \exp\{-\tfrac{1}{4}\pi^2(\kappa t/l^2)\} \tag{15}$$

for values of $\kappa t/l^2$ greater than about 0·1. If E represents at any instant of time that fraction of the total expected increment Δp_0 which still has to appear on the manometer, then $E = 1 - P = 1 \cdot 27 \exp\{-2 \cdot 46 \kappa t/l^2\}$.

It has been mentioned above that in these experiments $\kappa t/l^2 = 0 \cdot 90$. This corresponds to a theoretical value of E of 0·15 and implies that, if the theory were correct, the times were recorded when the manometer reached 85% of its final value instead of 90%, which can be considered a very good agreement.

That the flow rates obtained above are consistent with the model used, which assumed laminar viscous flow, can be shown as follows. The Reynolds number R, which determines the type of flow, is given by $R = 2av\rho/\eta$ where ρ the density and v the speed of flow, equal to $q/\pi a^2$. Also q is given by equation (1), so that

$$R = \frac{a^3 \rho}{4\eta^2} \left|\frac{\partial p}{\partial x}\right|. \tag{16}$$

Now $|\partial p/\partial x|$ has its maximum value at $x = l$, which is the open end of the tube. This can be seen by considering that $\partial p/\partial t$ is obviously of the same sign everywhere, and so therefore is $\partial^2 p/\partial x^2$, from equation (11). But the value of $|\partial p/\partial x|$ is zero at $x = 0$, so its value increases steadily up to $x = l$. To find its value approximately at $x = l$, consider the case of a semi-infinite pipe extending from $x = l$ to $x = -\infty$, since near $t = 0$, when the gradients are steepest, this case does not differ appreciably from the case of a finite pipe. The solution for such a pipe is (Carslaw and Jaeger 1948, p. 45)

$$p = \Delta p_0 \left[1 - \text{erf}\left\{\frac{x - l}{2(\kappa t)^{1/2}}\right\}\right]$$

and at $x = l$

$$\frac{\partial p}{\partial x} = \frac{\Delta p_0}{(\pi \kappa t)^{1/2}}. \tag{17}$$

The Reynolds number decreases to a value of about 10^3 appropriate to laminar flow, in a space of time given by

$$R = \frac{a^3 \rho}{4\eta^2} \frac{\Delta p_0}{(\pi \kappa t)^{1/2}} = 10^3,$$

or $$t = \left\{\frac{a^3 \rho}{4\eta^2} \frac{\Delta p_0}{(\pi \kappa)^{1/2}} \frac{1}{10^3}\right\}^2 \simeq (2 \cdot 53 \times 10^{-8}) \Delta^2 p_0 \text{ sec}$$

for $\frac{3}{16}$ in. bore tube. Here Δp_0 is measured in *pdl* ft^{-2}. This can alternatively be written

$$t > (6 \cdot 08 \times 10^{-4}) W^2 \text{ sec}$$

where W is the change in pressure in inches of water gauge. The flow will therefore be laminar everywhere for all but a negligible interval of time, since W will not generally be a large number.

In this theory the volume change in the manometer has been neglected as is justified for a liquid-in-glass manometer. This could be taken into account if necessary by taking in place of the boundary condition at $x = 0$ given in (9) a boundary condition which involved coupling with the equation of motion of the manometer.

3. Laboratory measurements of response time

As an experimental check of the law that the response time varies as the square of the length and to determine values of the time for a given diameter of tube, several series of measurements were made using a water-in-glass manometer. One limb of the manometer was connected by different lengths of Neoprene tubing of $\frac{3}{16}$ in. bore, up to the maximum length available of 1200 ft, to a source of pressure, the other limb being open to atmospheric pressure. This use of a manometer arises with pressure surveys along coal-mine roadways. The pressure was both applied to and removed from the measuring system using a two-way tap, one leg of which was connected to a static pressure tube

inserted into a ventilation duct, the other leg being open to atmospheric pressure. The manometer was thoroughly cleaned free of grease and dirt before and during the experiments.

The first observations were carried out with a constant pressure of 3 in. water gauge and the time taken for 90% of the change in pressure, from zero to 3 in. and from 3 in. to zero, was noted for various lengths of tube. Table 1 shows the results obtained, each entry being the average of five observations. Whether the pressure is applied or removed the response times are the same. The least squares analysis of the results gives $t = (l/220\cdot2)^{2\cdot07}$ when l is in feet and t in seconds.

Table 1. Response times for different lengths of tubing with constant pressure (3 in. w.g.)

Length of tubing (ft)	Time to reach 90% of pressure change (sec)	
	Pressure applied	Pressure removed
300	2·0	2·1
450	4·1	4·0
600	7·3	7·0
750	14·2	14·2
900	19·8	19·4
1050	25·6	25·4
1200	31·7	31·4

A further short series of observations was completed by applying different pressures, up to 8 in. water gauge, to a constant length of tubing, namely 1240 ft. The mean value of the observed 90% response times was 38 sec, which is close to the value of 36 sec to be expected from the above least squares analysis. There was no significant change in the time for different values of the pressure.

Table 2 gives the response times with one limb of the manometer connected to a source of pressure through 797 ft of the same tubing and also with the manometer moved to other positions along the tubing, namely 620, 400 and 0 ft, the rest of the length of tubing being attached to the other limb and at atmospheric pressure. Each tabulated figure is the average of 5 observations. Movement of the manometer along the tubing reduces the response time in accordance with the square law relationship, the open end of the manometer remaining at atmospheric pressure even though it has an increasing length of tubing attached to it.

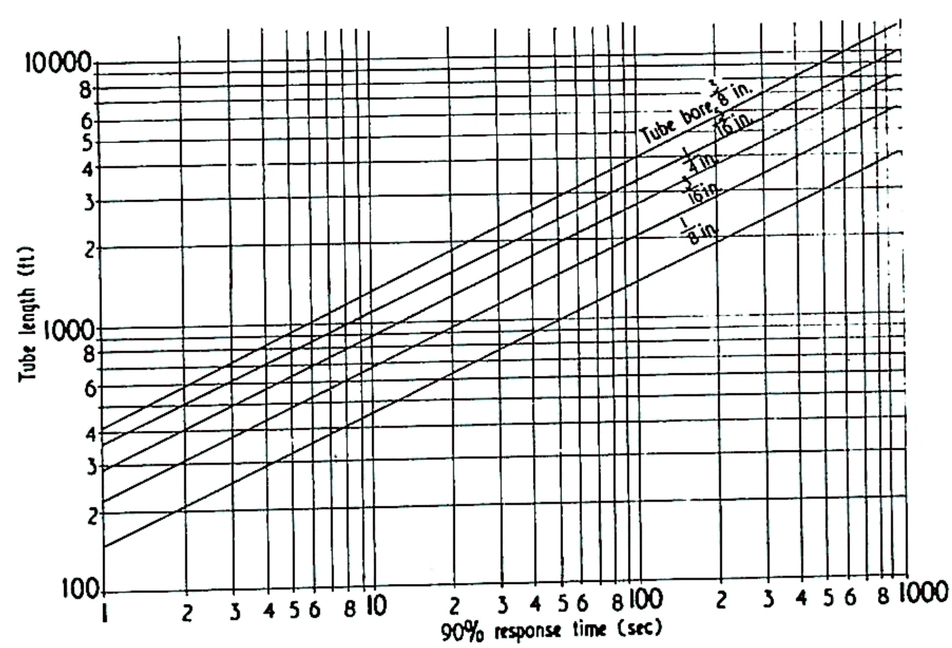
Table 2. Variations of manometer position along the tubing (pressure 0·56 in. w.g.)

Tubing length (ft)	90% response time (sec)	
	Pressure applied	Pressure removed
797	14·5	14·5
620	9·0	9·0
400	3·5	3·5
0	Instantaneous	Instantaneous

Although it seems unlikely that manometers with tubes of unequal length would be used extensively, a difficulty arises on which comment should be made here. If the pressures on both limbs change, the 90% response time will depend upon the actual, but unknown, values of the pressure changes as well as on the lengths of the tubes. This is illustrated by an extreme case, when there is a large change of pressure on the short tube and a small change on the long tube, the time to read 90% of the applied pressure difference being governed by the length of the short tube. If the pressures are interchanged the time is governed by the long tube. However, it is generally the case that the response time is not greater than that appropriate to the long tube. For manometer with equal tube lengths this difficulty does not arise, since the ratio of the pressures at the manometer each limb at any instant is the same as the ratio of the applied pressures.

Finally, the response times were measured at different pressures up to 8 in. water gauge using 600 ft of tubing, one case with the tube straight and in the other with the tube coiled but nevertheless free of kinks. Again there was no difference in the average values of the response times.

The figure has been drawn from the data given in table so that the reader can easily read the response times for any desired length up to 10000 ft of a number of different diameters of tubing.



Response time for different diameters of tubing.

4. Conclusions

For liquid-in-glass manometers the following conclusions are made:

(i) The 90% response time for long lengths of tubing connected in the open-end manner to a liquid-in-glass manometer varies as the square of the length and inversely as the square of the diameter of the bore of the tubing. The value of the 90% response time for 600 ft of $\frac{3}{16}$ in. bore tube is 8·0 seconds.

(ii) The response time does not depend on the value of the pressure or on whether the pressure is applied to or removed from the manometers. Neither does it matter whether the tubing is coiled or straight, provided it retains its natural cylindrical shape.

(iii) When a manometer has each limb connected by equal lengths of tubing to a source of pressure difference the response time is that for either length of tubing.

Acknowledgments

The authors wish to record their thanks to Mr. B. King for his assistance in the experimental work. The work described in this paper is published by permission of the Director General of Research.

Reference

CARSLAW, H. S., and JAEGER, J. C., 1948, *Conduction of Heat in Solids* (London: Oxford University Press).

LIBRARY PROGRAM—$RTT 321

Library Program

The following is a print-out of the library program $RTT 321, which is used to produce a numerical solution to the partial differential equation arising from the case study of Television Programme 2 and Radio Programme 5. The listing is followed by explanatory notes and a set of exercises.

```
10  PRINT "                 MANOMETER RESPONSE TIME"
20  PRINT "                 ------------------------"
30  PRINT
40  PRINT "**INPUT**"
50  PRINT
60  PRINT "INPUT LENGTH OF TUBE (FT) ";
70  INPUT L
80  PRINT
90  PRINT "INPUT BORE OF TUBE (INS)  ";
100  INPUT A
110  PRINT
120  PRINT "INPUT PARAMETER EPSILON    ";
130  INPUT E
140  PRINT
150  PRINT "NUMBER OF MESH POINTS      ";
160  INPUT N1
170  PRINT
180  PRINT "TIME STEP FOR IMPLICIT     ";
190  INPUT K
200  PRINT
210  READ Z,P0
220  DATA .000013,68000.
230  A=A/24
240  K1=A*A*P0/(8*Z)
250  P1=3.14159
260  DIM O[50],P[50],Q[50],A[50],B[50],C[50],D[50],W[50],S[50]
270  PRINT "**";
280  PRINT "PARAMETER KAPPA =";K1;
290  N2=N1-1
300  N3=N1-2
310  REM
320  REM   H= STEP LENGTH IN X-DIRECTION
330  REM
340  H=L/N2
350  K9=H*H/(2*K1)
360  REM
370  REM   M= NUMBER OF STEPS WITH EXPLICIT SCHEME
380  REM
390  M=INT(K/K9)+1
400  K9=K/M
410  REM
420  REM   R= .25 TIMES KAPPA TIMES MESH RATIO (K/H↑2)
430  REM
440  R=K9*K1/(4*H*H)
450  PRINT "TIME STEP FOR EXPLICIT METHOD =";K9
```

```
460  PRINT
470  PRINT "FOR EXPLICIT METHOD ONLY TYPE E OTHERWISE TYPE I";
480  INPUT A$
490  PRINT
500  PRINT "FOR COMPLETE PRINT-OUT TYPE P OTHERWISE TYPE N";
510  INPUT F$
520  PRINT
530  PRINT "**WORKING**"
540  PRINT
550  IF F$="N" THEN 640
560  PRINT "SOLUTIONS GIVEN AT X=0"
570  PRINT
580  PRINT "TIME                NUMERICAL        ANALYTICAL        'ERROR'"
590  PRINT "                    SOLUTION         SOLUTION"
600  PRINT
610  REM
620  REM      INITIAL CONDITIONS
630  REM
640  FOR I=1 TO N1
650  O[I]=0
660  NEXT I
670  O[N1]=.5
680  REM
690  REM      EXPLICIT SCHEME
700  REM
710  J=0
720  J=J+1
730  P[1]=O[1]+8*R*(1+E*O[1])*(O[2]-O[1])
740  P[N1]=1
750  FOR I=2 TO N2
760  P[I]=(O[I+1]-O[I-1])*(O[I+1]-O[I-1])
770  P[I]=O[I]+R*(E*P[I]+4*(1+E*O[I])*(O[I+1]-2*O[I]+O[I-1]))
780  NEXT I
790  T=K9*J
800  FOR I=1 TO N1
810  O[I]=P[I]
820  NEXT I
830  IF F$="N" THEN 860
840  GOSUB 1680
850  PRINT T,P[1],S1,S1-P[1]
860  IF P[1] >= .9 THEN 1480
870  REM
880  REM   HANDOVER FROM EXPLICIT TO IMPLICIT SCHEME
890  REM
900  IF A$="E" THEN 720
910  IF J=M THEN 940
920  IF J=M*2 THEN 980
930  GOTO 720
940  FOR I=1 TO N1
950  Q[I]=O[I]
960  NEXT I
970  GOTO 720
980  FOR I=1 TO N1
990  O[I]=Q[I]
1000  NEXT I
1010  PRINT
1020  PRINT "EXPLICIT SCHEME FINISHED AFTER";J;"STEPS"
1030  PRINT
1040  REM
```

```
1050  REM         IMPLICIT SCHEME
1060  REM
1070  J=2
1080  REM
1090  REM  R= TWO-THIRDS KAPPA TIMES MESH-RATIO
1100  REM
1110  R=2*K*K1*N2*N2/(3*L*L)
1120  J=J+1
1130  A3=R*(2+E*(P[1]+P[2]))
1140  B[1]=1+A3
1150  C[1]=A3
1160  D[1]=O[1]+A3*(P[2]+O[2]-P[1]-O[1])
1170  FOR I=2 TO N3
1180  A1=R*(1+E*.5*(P[I]+P[I-1]))
1190  A2=R*(1+E*.5*(P[I+1]+P[I]))
1200  A3=A1+A2
1210  A[I]=A1
1220  B[I]=1+A3
1230  C[I]=A2
1240  D[I]=O[I]+A1*(P[I-1]+O[I-1])+A2*(P[I+1]+O[I+1])-A3*(P[I]+O[
1250  NEXT I
1260  A1=R*(1+E*.5*(P[N2]+P[N3]))
1270  A2=R*(1+E*.5*(1+P[N2]))
1280  A3=A1+A2
1290  A[N2]=A1
1300  B[N2]=1+A3
1310  D[N2]=O[N2]+A1*(P[N3]+O[N3])+3*A2-A3*(P[N2]+O[N2])
1320  T=J*K
1330  FOR I=1 TO N1
1340  O[I]=P[I]
1350  NEXT I
1360  GOSUB 1530
1370  IF F$="N" THEN 1400
1380  GOSUB 1680
1390  PRINT T,P[1],S1,S1-P[1]
1400  IF P[1]<.9 THEN 1120
1410  T=J*K
1420  PRINT "IMPLICIT SCHEME FINISHED AFTER";J;"STEPS"
1430  PRINT
1440  PRINT
1450  PRINT "**SOLUTION**"
1460  PRINT
1470  PRINT
1480  PRINT "RESPONSE TIME =";T;"SECS'
1490  STOP
1500  REM
1510  REM TO SOLVE A TRIDIAGONAL SYSTEM OF EQUATIONS
1520  REM
1530  W[1]=B[1]
1540  S[1]=D[1]
1550  FOR I=2 TO N2
1560  Z=A[I]/W[I-1]
1570  W[I]=B[I]-C[I-1]*Z
1580  S[I]=D[I]+S[I-1]*Z
1590  NEXT I
1600  P[N2]=S[N2]/W[N2]
1610  FOR I=N3 TO 1 STEP -1
1620  P[I]=(S[I]+C[I]*P[I+1])/W[I]
1630  NEXT I
1640  RETURN
```

```
1650  REM
1660  REM   TO CALCULATE THE ANALYTICAL SOLUTION TO THE LINEAR PROBL
1670  REM
1680  S1=Ø
1690  N=Ø
1700  S2=(-1)↑N*EXP(-(N+.5)*(N+.5)*P1*P1*K1*T/(L*L))/(N+.5
1710  S1=S1+S2
1720  IF ABS(S2)<1.E-Ø7 THEN 175Ø
1730  N=N+1
1740  GOTO 17ØØ
1750  S1=1-S1*2/P1
1760  RETURN
1770  END
```

Program $RTT 321

1 Objective

To solve numerically the problem

$$\frac{\partial P}{\partial t} = \kappa \frac{\partial}{\partial x}\left\{(1 + \varepsilon P)\frac{\partial P}{\partial x}\right\} \qquad 0 < x < l, \quad t > 0,$$

$$P(x, 0) = 0 \qquad 0 \leqslant x \leqslant l,$$

$$P(l, t) = 1 \qquad t > 0,$$

$$\frac{\partial P}{\partial x}(0, t) = 0 \qquad t > 0.$$

In particular, to find the value of t for which $P(0, t)$ first becomes greater than 0.9. This value of t is known as the *response time*. The problem models a long thin tube of length l which transmits a differential air pressure down its length. The finite-difference method is used and the region is covered with a rectangular mesh.

2 Input

(i) length of tube (ft)	program symbol L
(ii) bore of tube (in)	A
(iii) parameter epsilon	E
(iv) number of mesh points in x-direction	N1
(v) time step for implicit method	K

3 Other Data (input via a DATA statement)

(i) coefficient of viscosity of air (η) — program symbol Z, value 1.3×10^{-5},

(ii) pressure under undisturbed conditions (p_0) — symbol PØ, value 6.8×10^{4}.

4 Options

(i) Use either an explicit or an implicit scheme (Type E or I).

(ii) Print out response time only or get fuller print-out (Type N or P).

5 Intermediate Calculations and Informative Print-out

(i) Evaluation and print-out of parameter κ (kappa) given by

$$\kappa = a^2 p_0/8\eta,$$

where a is the radius of the tube.

(ii) Calculation of optimum time step for explicit scheme.

(See Section 9, Implicit Scheme.)

6 Initial Conditions

If there are $N1$ mesh points in the x-direction then

$P_{i,0} = 0 \qquad i = 1, 2, \ldots, N1 - 1,$

$P_{N1,0} = 0.5 \qquad$ (where P is now being used as a symbol to describe the finite-difference solution for the pressure difference as well as the analytic solution).

(See exercise 6.)

7 Output

If full print-out is required then the numerical solution of the differential equation is printed out at the point $x = 0$ at each time step together with the analytical solution to the linear equation ($\varepsilon = 0$) at the same point. The difference ('ERROR') between these two values is also given. The final output is the response time.

8 Explicit Scheme

The explicit finite-difference replacement of the nonlinear equation is obtained from the form of the equation

$$\frac{\partial P}{\partial t} = \kappa\left[\varepsilon\left(\frac{\partial P}{\partial x}\right)^2 + (1 + \varepsilon P)\frac{\partial^2 P}{\partial x^2}\right],$$

from which we obtain

$$P_{i,j+1} = P_{i,j} + R\{\varepsilon(P_{i+1,j} - P_{i-1,j})^2 + 4(1 + \varepsilon P_{i,j})(P_{i+1,j} - 2P_{i,j} + P_{i-1,j})\},$$

where

$$R = \kappa k/(4h^2).$$

For stability we require $\kappa r \leqslant \frac{1}{2}$ where $r = k/h^2$.

(See also Section 9, Implicit Scheme.)

9 Implicit Scheme

The implicit scheme for the equation

$$b\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(U)\frac{\partial U}{\partial x}\right] \qquad a(U) > 0, \quad b > 0, \tag{1}$$

is given by

$$\begin{aligned} b(u_{i,j+1} - u_{i,j-1}) = \tfrac{2}{3}r[\alpha^+\{(u_{i+1,j+1} - u_{i,j+1}) + (u_{i+1,j} - u_{i,j}) \\ + (u_{i+1,j-1} - u_{i,j-1})\} - \alpha^-\{(u_{i,j+1} - u_{i-1,j+1}) \\ + (u_{i,j} - u_{i-1,j}) + (u_{i,j-1} - u_{i-1,j-1})\}],\end{aligned}$$

where

$$\alpha^+ = a\left(\frac{u_{i+1,j} + u_{i,j}}{2}\right), \quad \alpha^- = a\left(\frac{u_{i,j} + u_{i-1,j}}{2}\right),$$

and r is the mesh ratio k/h^2. The scheme is a three-level scheme since it involves values of u on the three time levels $j + 1, j$ and $j - 1$.

For the problem

$$\begin{aligned}
&\frac{1}{\kappa}\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left((1+\varepsilon P)\frac{\partial P}{\partial x}\right) && 0 < x < l, \quad t > 0, \\
&P(x, 0) \;\; = 0 && 0 \leqslant x \leqslant l, \\
&P(l, t) \;\;\; = 1 && t > 0, \\
&\frac{\partial P}{\partial x}(0, t) = 0 && t > 0,
\end{aligned} \tag{2}$$

we see that in the notation of equation (1), $a(P) = 1 + \varepsilon P$, $b = 1/\kappa$, and so

$$\alpha^+ = 1 + \tfrac{1}{2}\varepsilon(P_{i+1,j} + P_{i,j}),$$
$$\alpha^- = 1 + \tfrac{1}{2}\varepsilon(P_{i,j} + P_{i-1,j}).$$

It is more efficient in the computer program to represent the values of P along the $(j-1)$th level by the array O[I], I = 1, 2, . . . , N1, and along the jth level by the array P[I], I = 1, 2, . . . , N1, (where we have assumed $N1$ mesh points in the x-direction) than to use a two-dimensional array. In this notation the implicit scheme becomes

$$\begin{aligned}
-R\alpha^+P_{i+1,j+1} + (1 + R(\alpha^+ + \alpha^-))P_{i,j+1} - R\alpha^-P_{i-1,j+1} &= R\alpha^+\mathrm{P}[\mathrm{I}+1] \\
&\quad - R(\alpha^+ + \alpha^-)\mathrm{P}[\mathrm{I}] + R\alpha^-\mathrm{P}[\mathrm{I}-1] + R\alpha^+\mathrm{O}[\mathrm{I}+1] \\
&\quad + (1 - R(\alpha^+ + \alpha^-))\mathrm{O}[\mathrm{I}] + R\alpha^-\mathrm{O}[\mathrm{I}-1],
\end{aligned} \tag{3}$$

where $R = \frac{2}{3}\kappa r$ and

$$\alpha^+ = 1 + \tfrac{1}{2}\varepsilon(\mathrm{P}[\mathrm{I}+1] + \mathrm{P}[\mathrm{I}]),$$
$$\alpha^- = 1 + \tfrac{1}{2}\varepsilon(\mathrm{P}[\mathrm{I}] + \mathrm{P}[\mathrm{I}-1]).$$

We can see from equation (3) that the implicit scheme involves three unknowns $P_{i+1,j+1}$, $P_{i,j+1}$ and $P_{i-1,j+1}$, and therefore gives rise to a tridiagonal system of equations. These can be solved by the recurrence relation form of Gauss elimination as given in *S*: pages 20 to 22. There, the tridiagonal set of equations have the form

$$\begin{aligned}
b_1u_1 - c_1u_2 \qquad\qquad\qquad\qquad &= d_1, \\
-a_2u_1 + b_2u_2 - c_2u_3 \qquad\qquad &= d_2, \\
\cdots \qquad\qquad\qquad\qquad &\;\;\vdots \\
-a_{N-1}u_{N-2} + b_{N-1}u_{N-1} &= d_{N-1}.
\end{aligned}$$

In the computer program we have used the same notation (where the lower case letters have naturally had to be changed to upper case). Thus in general

$$\mathrm{B}[\mathrm{I}] = 1 + R(\alpha^+ + \alpha^-),$$
$$\mathrm{A}[\mathrm{I}] = R\alpha^-,$$
$$\mathrm{C}[\mathrm{I}] = R\alpha^+,$$

and D[I] is given by the right-hand side of equation (3). The solution of the system of equations is stored in the array P after the contents of P have been transferred to the array O. In this way the solution along the jth level is always in the array P and the solution along the $(j-1)$th level is always in the array O.

Since the implicit scheme is a three-level scheme we need values along $j = 0$ and $j = 1$ before it can be used. Along $j = 0$ we have initial values already given. For values along $j = 1$ we use the explicit scheme. However, this process involves some careful computation as we shall now illustrate.

The explicit scheme has a restriction on the size of time step that can be used (in order to retain stability), namely

$$\kappa r \leqslant \tfrac{1}{2}, \quad r = k/h^2,$$

or

$$k \leqslant \frac{h^2}{2\kappa}. \qquad (4)$$

The object of using the implicit scheme is that we can use larger time steps than with the explicit scheme and thereby reduce the amount of computation required. This implies that the time step we use in the explicit scheme will be different (smaller) from that used in the implicit scheme. Therefore, to be able to use the explicit scheme to "start off" the implicit scheme the step length of the explicit scheme has to be such that the step length of the implicit scheme is a multiple of it. The computer program therefore allows you to specify any value for the implicit time step (call it k_I) and then calculates the largest explicit time step (call it k_E) such that

$$k_I = Mk_E \qquad \text{where } M \text{ is an integer}$$

and such that k_E satisfies equation (4).

In fact we choose to use the explicit scheme to provide values along $j = k_I$ and $j = 2k_I$ (rather than just $j = k_I$).

This is done to ensure that the effects of the discontinuity in the initial and boundary data at $(l, 0)$, which have been minimized by initially using a finite-difference scheme with a small time step, are not allowed to recur at the start of the implicit scheme.

10 Exercises

The first four exercises discuss aspects of the modelling.

1 What are the response times for tubes of $\frac{1}{8}$ in bore which are

(a) 300 ft long?

(b) 600 ft long?

Use the implicit scheme with 7 mesh points in the x-direction and take $\varepsilon = 0.01$.

Hint: to reduce computation time use a step length which requires only about 20 steps with the implicit scheme. You can estimate the required value by referring to the graph of response time against tube length given in the paper by Jones and Jordan.

2 Repeat exercise 1 for tubes of $\frac{3}{8}$ in bore.

3 What is the effect on the results of exercises 1 and 2 if you put $\varepsilon = 0.1$ (the maximum value for ε in practice)?

4 Is the linear case ($\varepsilon = 0$) a good approximation to the given problem for all practical values of ε?

Hint: compare the results of exercises 1, 2 and 3.

The following exercises discuss aspects of the numerical analysis.

5 Put L = 300, A = 0.1, E = 0, N1 = 7, K = 0.5 and compare the times taken by the explicit and implicit schemes to evaluate the response time.

6 Choose some typical input values for the program and opt for the full print-out. Now change statement 670 to

```
670     O[N1] = 0
```

and run the program with the same data. Compare the 'ERROR' columns in the two print-outs and say why we choose to take $P_{N1,0} = 0.5$ when the true initial conditions of the problem specify $P_{N1,0} = 0$.

Hint: what is peculiar about the point $(l, 0)$?

Note Before going on to other exercises make sure you change statement 670 back to

```
670     O[N1] = .5
```

M321 R6

RADIO PROGRAMME 6

Broadcast Notes

CONVERGENCE

This programme studies the circumstances under which the solution of a finite-difference scheme converges to the analytic solution of the approximated partial differential equation as the mesh spacings are reduced. The approaches to convergence for initial-boundary and pure boundary value problems are contrasted by investigating a simple example from each category. The presenter is Peter Thomas. Reference is made to the following notes.

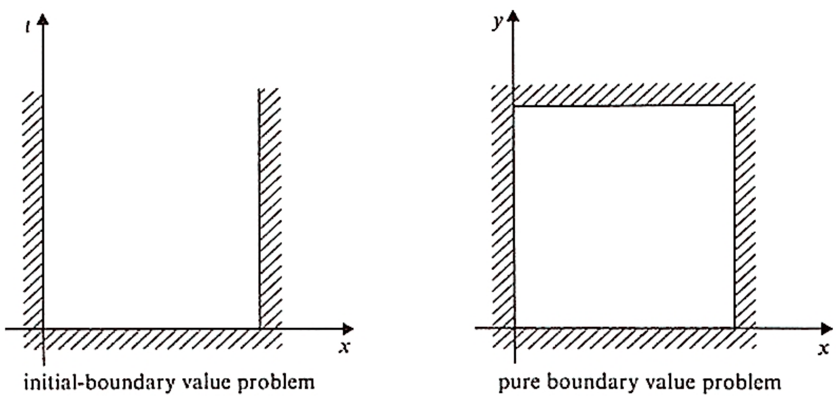


Fig. 1

Consider the initial-boundary value problem

$$\begin{aligned} &\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} && 0 < x < 1, t > 0, \\ &U(x,0) = f(x) && 0 < x < 1, \\ &U(0,t) = U(1,t) = 0 && t \geqslant 0. \end{aligned} \tag{1}$$

The partial differential equation can be approximated by the simple explicit finite-difference scheme

$$u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \tag{2}$$

where $r = k/h^2$, h and k being the mesh spacings in the x- and t-directions respectively. The finite-difference approximation to the problem (1) is then

$$\begin{aligned} &u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) && i = 1, 2, \ldots, N-1; j = 1, 2, \ldots, \\ &u_{i,0} = f(ih) && i = 1, 2, \ldots, N-1, \\ &u_{0,j} = u_{N,j} = 0 && j = 0, 1, 2, \ldots, \end{aligned} \tag{3}$$

where $Nh = 1$. This can be put in matrix form as

$$\begin{aligned} &\mathbf{u}_{j+1} = A\mathbf{u}_j \qquad j = 0, 1, 2, \ldots, \\ &\mathbf{u}_0 = \mathbf{f}, \end{aligned} \tag{4}$$

where $\mathbf{u}_j$ is the column vector $(u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})$, $\mathbf{f} = (f(h), f(2h), \ldots, f[(N-1)h])$ and A is a square matrix of order $N-1$, namely

$$A = \begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & & \cdot \;\cdot\; \cdot & & \\ & & r & 1-2r & r \\ & & & r & 1-2r \end{bmatrix}.$$

The partial differential equation in (1) could alternately be approximated by the Crank–Nicolson implicit scheme,

$$-ru_{i-1,j+1} + (2 + 2r)u_{i,j+1} - ru_{i+1,j+1}$$
$$= ru_{i-1,j} + (2 - 2r)u_{i,j} + ru_{i+1,j}, \quad (5)$$

with which the matrix form of the problem becomes

$$\begin{aligned} B\mathbf{u}_{j+1} &= C\mathbf{u}_j \qquad j = 0, 1, 2, \ldots, \\ \mathbf{u}_0 &= \mathbf{f} \end{aligned} \qquad (6)$$

where $\mathbf{u}_j$ and $\mathbf{f}$ are defined as in the explicit case, B and C are square matrices of order $N - 1$ given by

$$B = \begin{bmatrix} 2+2r & -r & & & \\ -r & 2+2r & -r & & \\ & \cdot & \cdot & \cdot & \\ & & -r & 2+2r & -r \\ & & & -r & 2+2r \end{bmatrix},$$

$$C = \begin{bmatrix} 2-2r & r & & & \\ r & 2-2r & r & & \\ & \cdot & \cdot & \cdot & \\ & & r & 2-2r & r \\ & & & r & 2-2r \end{bmatrix}.$$

If B is non-singular then equations (6) can be solved uniquely and we can write

$$\begin{aligned} \mathbf{u}_{j+1} &= D\mathbf{u}_j \qquad j = 0, 1, 2, \ldots, \\ \mathbf{u}_0 &= \mathbf{f} \end{aligned} \qquad (7)$$

where $D = B^{-1}C$. This is of the same form as equations (4), so that if we drop the definition of D as a product of two known matrices B^{-1} and C we can regard (7) as representing both the explicit and the implicit method. From (7) we obtain

$$\begin{aligned} \mathbf{u}_0 &= \mathbf{f}, \\ \mathbf{u}_1 &= D\mathbf{u}_0, \\ \mathbf{u}_2 &= D\mathbf{u}_1 = D(D\mathbf{u}_0) = D^2\mathbf{u}_0, \\ &\vdots \\ \mathbf{u}_n &= D\mathbf{u}_{n-1} = D(D\mathbf{u}_{n-2}) = \ldots = D^n\mathbf{u}_0. \end{aligned} \qquad (8)$$

Suppose $U(x, t)$ is the true solution of problem (1), and that $\mathbf{U}_j = (U(h, jk), U(2h, jk), \ldots, U[(N-1)h, jk])$. Since $U(ih, jk)$ differs from $u_{i,j}$ in general, $\mathbf{U}_{j+1} - D\mathbf{U}_j = \mathbf{T}_j \neq \mathbf{0}$, where $\mathbf{T}_j$ represents the errors incurred by using the finite-difference scheme to progress from the jth to the $(j + 1)$th time level.

We have

$$\mathbf{U}_{j+1} = D\mathbf{U}_j + \mathbf{T}_j \qquad j = 0, 1, 2, \ldots \qquad (9)$$

It also follows from (1) that

$$\mathbf{U}_0 = \mathbf{f}. \qquad (10)$$

Writing out (9) explicitly,

$$\begin{aligned} \mathbf{U}_1 &= D\mathbf{U}_0 + \mathbf{T}_0 \\ \mathbf{U}_2 &= D\mathbf{U}_1 + \mathbf{T}_1 = D(D\mathbf{U}_0 + \mathbf{T}_0) + \mathbf{T}_1 = D^2\mathbf{U}_0 + D\mathbf{T}_0 + \mathbf{T}_1, \\ &\vdots \\ \mathbf{U}_n &= D^n\mathbf{U}_0 + \sum_{j=0}^{n-1} D^{n-1-j}\mathbf{T}_j. \end{aligned} \qquad (11)$$

Combining the second equation of (7) with (10),

$$\mathbf{u}_0 = \mathbf{U}_0 = \mathbf{f}, \tag{12}$$

so that

$$\mathbf{U}_n - \mathbf{u}_n = \sum_{j=0}^{n-1} D^{n-1-j}\mathbf{T}_j \qquad n = 1, 2, \ldots. \tag{13}$$

The finite-difference scheme is defined to be convergent to the true solution on the fixed time level t if

$$\lim_{\substack{h,k \to 0 \\ nk = t}} \|\mathbf{U}_n - \mathbf{u}_n\| = 0, \tag{14}$$

where $\| \ \|$ denotes the norm of the vector (a measure of its size). It follows from (13) that convergence is guaranteed if

$$\lim_{\substack{h,k \to 0 \\ nk = t}} \left\| \sum_{j=0}^{n-1} D^{n-1-j}\mathbf{T}_j \right\| = 0. \tag{15}$$

If it takes n steps of length k to reach the fixed time level t (as we have assumed above), then

$$t = nk. \tag{16}$$

Keeping t fixed and letting $k \to 0$ means that $n \to \infty$, so that (15) involves an infinite sum. The convergence condition will be satisfied if both $\|\mathbf{T}_j\| \to 0 \; \forall j$ as $h, k \to 0$ (consistency), and all the powers of the matrix D are bounded (in some sense not defined here; it is sufficient for the spectral radius of D to be less than one). A scheme for which all powers of D are bounded is said to be stable. Thus consistency and stability together imply convergence. This illustrates Lax's Theorem.

Consider next the pure boundary value problem

$$\begin{aligned} &\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 && 0 < x < 1, 0 < y < 1, \\ &U(x, 0) = f_1(x) && 0 \leqslant x \leqslant 1, \\ &U(x, 1) = f_2(x) && 0 \leqslant x \leqslant 1, \\ &U(0, y) = f_3(y) && 0 \leqslant y \leqslant 1, \\ &U(1, y) = f_4(y) && 0 \leqslant y \leqslant 1. \end{aligned} \tag{17}$$

The unit square is covered with a square mesh. The mesh spacing is $h = 1/N$.

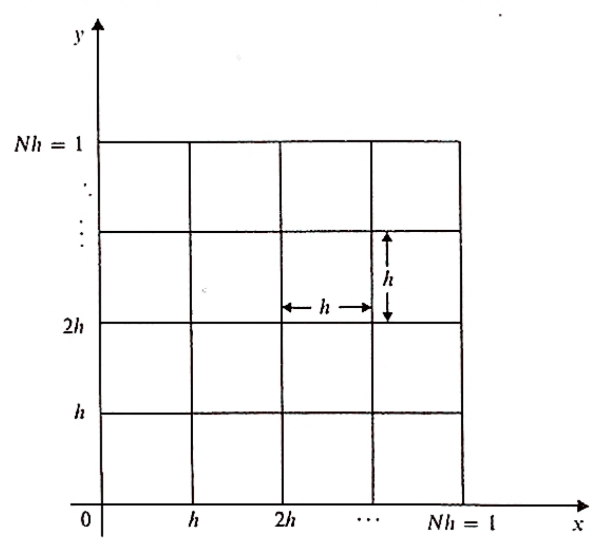


Fig. 2

The partial differential equation is replaced by the five-point formula

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 0. \tag{18}$$

By writing this down for every internal mesh point we get the matrix equation

$$A\mathbf{u} = \mathbf{b}, \tag{19}$$

where $\mathbf{u} = (u_{1,1}, u_{1,2}, \ldots, u_{1,N-1}, u_{2,1}, \ldots, u_{N-1,N-1})$, $\mathbf{b}$ has elements determined by the boundary values, and A is a square matrix of order $(N-1)^2$ which has the partitioned form

$$A = \begin{bmatrix} B & I & & & \\ I & B & I & & \\ & \cdot & \cdot & \cdot & \\ & & I & B & I \\ & & & I & B \end{bmatrix}, \tag{20}$$

I being the identity matrix of order $N-1$ and B the matrix

$$B = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix}.$$

Since all the mesh values are to be computed simultaneously the concept of stability, which we used when investigating an initial-boundary value problem, is not meaningful here. We must therefore approach convergence differently.

It is shown in *Unit 11* that when the five-point formula (18) is applied to the Dirichlet problem for Laplace's equation in a rectangle it gives a unique solution. It is also shown that if v is any function defined on the set of mesh points in some rectangular region D, one of whose sides has length a, then

$$\max_{D_\Delta} |v| \leqslant \max_{C_\Delta} |v| + \tfrac{1}{2}a^2 \max_{D_\Delta} |Lv|, \tag{21}$$

where D_Δ is the set of mesh points interior to D, C_Δ the mesh points on the boundary of D, and L the five-point finite-difference operator (i.e. the left-hand side of (18) could be written as $Lu_{i,j}$).

As the vector $\mathbf{u}$ which satisfies equation (19) is only an approximation to the true solution $\mathbf{U}$ (as in the previous case) we can find a non-zero vector $\mathbf{T}$ such that

$$A\mathbf{U} = \mathbf{b} + \mathbf{T}. \tag{22}$$

Subtracting (19) from this,

$$A(\mathbf{U} - \mathbf{u}) = \mathbf{T}, \tag{23}$$

and since we know the solution of (19) to be unique the matrix A can be inverted, giving

$$\mathbf{U} - \mathbf{u} = A^{-1}\mathbf{T}. \tag{24}$$

For convergence, we require that

$$\lim_{h \to 0} \|\mathbf{U} - \mathbf{u}\| = 0, \tag{25}$$

and from (24) this will hold if and only if

$$\lim_{h \to 0} \|A^{-1}\mathbf{T}\| = 0. \tag{26}$$

Analogously to the initial-boundary value case there are two conditions which when satisfied together will ensure that (26) holds. The first is that the elements of $\mathbf{T}$ (which are in fact the local truncation errors) must tend to zero as $h \to 0$. The second is that A must be non-singular. The inequality (21) can be used to obtain an upper bound on the global error of the finite-difference scheme. This is in fact of the same order of magnitude as the local truncation errors.

Problem

Numerical methods can often reveal or illustrate results of mathematical analysis. One such result is that a problem requiring the solution of a hyperbolic equation in a closed region and the attainment of given values on its boundary is not properly posed. Illustrate this by trying to solve numerically the problem

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} = 0 \qquad 0 < x < 1,\ 0 < y < 1,$$

$$U(x, 0) = f_1(x) \qquad 0 \leqslant x \leqslant 1,$$

$$U(x, 1) = f_2(x) \qquad 0 \leqslant x \leqslant 1,$$

$$U(0, y) = f_3(y) \qquad 0 \leqslant y \leqslant 1,$$

$$U(1, y) = f_4(y) \qquad 0 \leqslant y \leqslant 1.$$

Take a square mesh with spacing $\frac{1}{3}$ and replace the second derivatives with the usual central difference expressions. The inverse of the matrix which represents the application of the finite-difference scheme at each interior mesh point must then be found.

M321 TV3

TELEVISION PROGRAMME 3

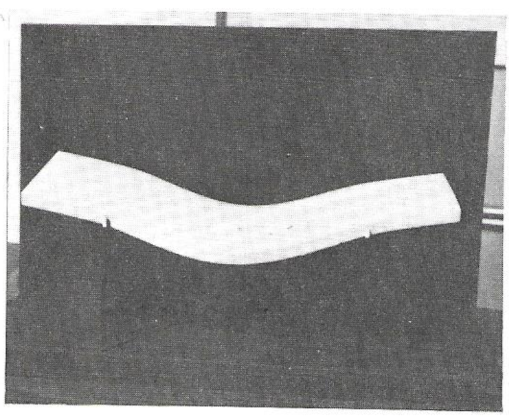
Broadcast Notes

SHALLOW WATER WAVES

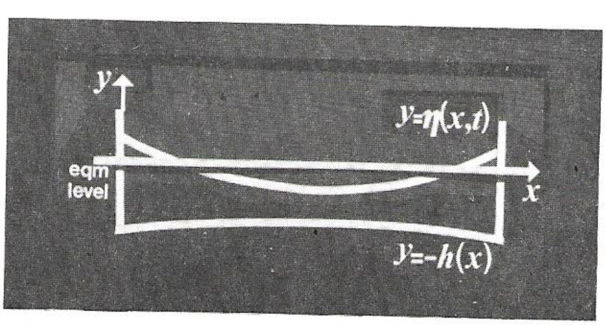
This programme investigates the behaviour of standing shallow water waves in a tank with vertical ends and constant longitudinal cross-section. The exact period of the simplest such wave is found for the case of a tank with horizontal bottom. Results from *Unit 13* are then used to estimate the corresponding quantity for a tank with sloping bottom.

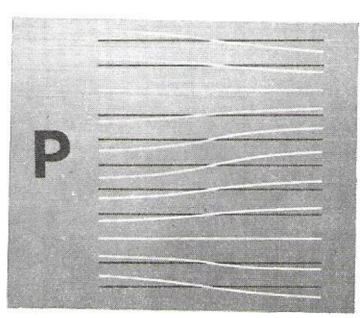
Introduction

Ralph Smith opens the programme by displaying water waves in a glass tank. The simplest such motions are the *stationary* or *standing waves*, which correspond to the eigenfunctions of a Sturm-Liouville system derived (later in the programme) from the shallow water wave equation. The essential feature of a standing water wave is that the surface contains points which neither rise nor fall with time (the *nodes*). A model shows the behaviour of such a surface with just two nodes.



For the purpose of describing mathematically the water surface motion in a tank with vertical ends and constant longitudinal cross-section we take the y-axis to coincide with the left-hand end of the tank and the x-axis to lie along the equilibrium level of the surface. The configuration of the surface in motion is given by $y = \eta(x, t)$, and the shape of the bottom of the tank (which will not in general be horizontal or even flat) is described by $y = -h(x)$.





The simplest standing wave has only one node, the surface in this case undergoing a slopping motion. We concentrate on this motion (which corresponds to the first eigenfunction of the associated Sturm-Liouville system) for the rest of the programme, seeking to find its period P.

The discussion so far has been in terms of the surface shape $\eta(x, t)$; it is in fact easier to deal with the equation for the horizontal fluid velocity $u(x, t)$ and this is the dependent variable used hereafter. u and η are connected by the two equations

$$\begin{aligned} u_t &= -g\eta_x, \\ (hu)_x &= -\eta_t, \end{aligned} \tag{1}$$

where g is the acceleration due to gravity and subscripts are once more being used to denote partial differentiation with respect to the subscripted variable (see Appendix for a derivation of these equations).

The horizontal fluid motion at the surface of the water in the studio tank is demonstrated with the aid of markers.

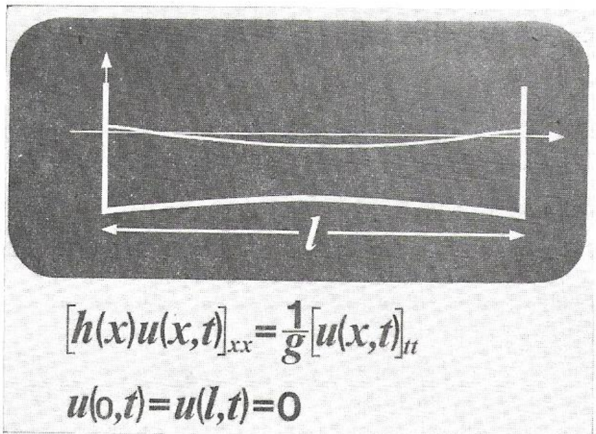
The Shallow Water Wave Equation

Dominic Jordan introduces the shallow water wave equation

$$(hu)_{xx} = \frac{1}{g}u_{tt}, \tag{2}$$

which is obtained by eliminating η between the two equations (1) above. This equation describes all possible motions within the limitations of the model. Since h is not in general a constant the equation is more complicated than the wave equation for (say) a uniform string. The horizontal fluid velocity is clearly zero at the ends of the tank, so if l is the tank's length we have the boundary conditions

$$u(0, t) = u(l, t) = 0. \tag{3}$$



Equations (2) and (3) together lead to an eigenvalue problem if we try and solve by separation of variables. Putting

$$u(x, t) = X(x)T(t) \tag{4}$$

effectively restricts our attention to the standing wave solutions, as is demonstrated by a piece of film. Solutions of this form always have their nodes, maxima and minima (with respect to variation of x) at fixed places, keeping the same basic shape at all times; only the amplitude varies.

Substitution of (4) into (2) and (3) gives

$$\begin{aligned} (hX)'' + \lambda X &= 0, \\ X(0) = X(l) &= 0, \end{aligned} \tag{5}$$

together with

$$T'' + \lambda g T = 0, \tag{6}$$

where λ is the separation constant.

Tank With Horizontal Bottom

We consider first the case where $h(x) \equiv H$, a constant. This corresponds to a tank with horizontal bottom. Equations (5) can then be written as

$$\begin{aligned} &X'' + \frac{\lambda}{H} X = 0, \\ &X(0) = X(l) = 0. \end{aligned} \tag{7}$$

Ralph Smith points out that the solutions (i.e. eigenfunctions) of this system correspond to standing waves in the tank. The eigenvalues of the system are

$$\lambda_n = n^2\pi^2 H/l^2 \qquad n = 1, 2, \ldots, \tag{8}$$

and the corresponding eigenfunctions are

$$X_n(x) = \sin(n\pi x/l) \qquad n = 1, 2, \ldots \tag{9}$$

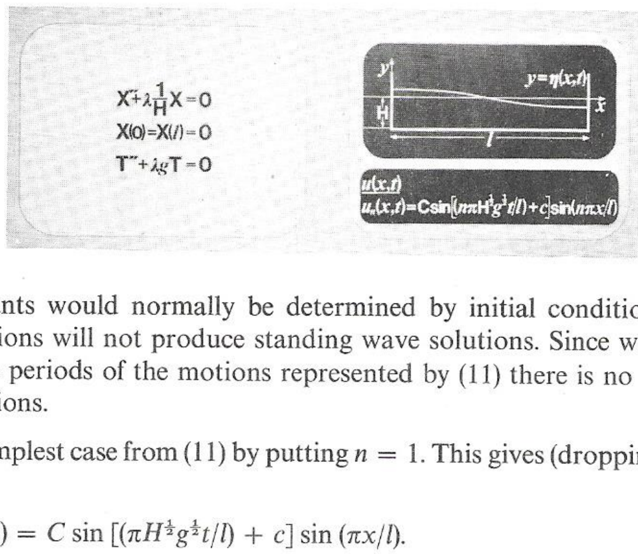
Solving equation (6) then gives

$$T_n(t) = C_n \sin\left[(n\pi H^{\frac{1}{2}} g^{\frac{1}{2}} t/l) + c_n\right] \qquad n = 1, 2, \ldots, \tag{10}$$

so that

$$u_n(x, t) = C_n \sin\left[(n\pi H^{\frac{1}{2}} g^{\frac{1}{2}} t/l) + c_n\right] \sin(n\pi x/l) \qquad n = 1, 2, \ldots, \tag{11}$$

is a set of separated solutions of equation (2) satisfying the boundary conditions (3). C_n and c_n are arbitrary constants. C_n is the amplitude of the motion at a maximum or minimum of X_n, and c_n depends upon what stage of the cycle has been reached at $t = 0$.



These constants would normally be determined by initial conditions, but typical initial conditions will not produce standing wave solutions. Since we are interested mainly in the periods of the motions represented by (11) there is no need to specify initial conditions.

We get the simplest case from (11) by putting $n = 1$. This gives (dropping the subscript throughout)

$$u(x, t) = C \sin\left[(\pi H^{\frac{1}{2}} g^{\frac{1}{2}} t/l) + c\right] \sin(\pi x/l). \tag{12}$$

A film shows the motion of the velocity profile with time in this case and the corresponding behaviour of the water surface (the simple slopping motion described earlier). It follows from (1) that the shape of the surface is given by

$$\eta(x, t) = C(H/g)^{\frac{1}{2}} \cos\left[(\pi H^{\frac{1}{2}} g^{\frac{1}{2}} t/l) + c\right] \cos(\pi x/l) \tag{13}$$

(see Appendix for details). The period of the motion is seen to be

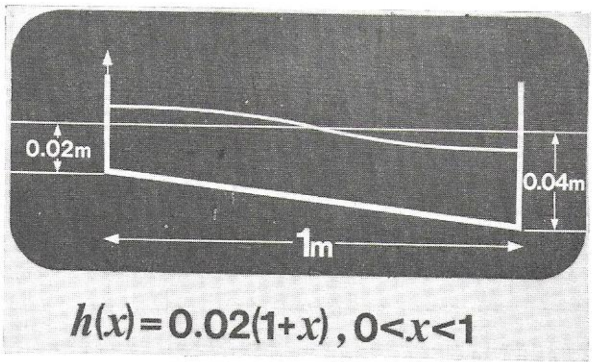
$$P = \frac{2\pi l}{\pi H^{\frac{1}{2}} g^{\frac{1}{2}}} = \frac{2l}{H^{\frac{1}{2}} g^{\frac{1}{2}}}. \tag{14}$$

Assigning typical values $l = 1\text{m}$, $H = 0.04\text{m}$, $g \simeq 10\text{ms}^{-2}$, we find P to be somewhere in the region of three seconds.

Tank With Sloping Bottom

Dominic Jordan considers the more complex situation which arises if the bottom of the tank is not horizontal. The equilibrium depth is taken to vary linearly from 2cm at one end to 4cm at the other, and it is supposed that the tank is 1m long. The depth function is therefore

$$h(x) = 0.02(1 + x) \qquad 0 \leqslant x \leqslant 1. \tag{15}$$



The situation is modelled by equations (5) (with $l = 1$) and (6) together with (15), assuming once again that attention is being restricted to the standing wave solutions. In order to tackle (5) using the methods of *Unit 13* the differential equation must be put into self-adjoint form, i.e. into the form

$$(pX')' - qX + \lambda\rho X = 0. \tag{16}$$

We find $p = h^2, q = -hh'', \rho = h$, so that the differential equation becomes

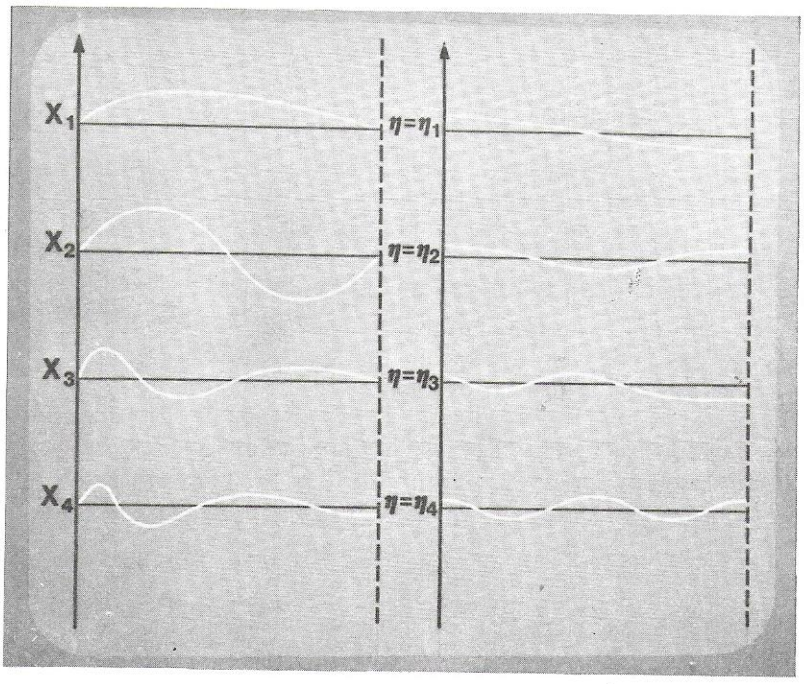
$$(h^2X')' + hh''X + \lambda hX = 0. \tag{17}$$

Substitution for h from (15) gives the Sturm–Liouville system

$$\begin{aligned}&[(1 + x)^2X']' + \lambda.50(1 + x)X = 0 \qquad 0 < x < 1,\\ &X(0) = X(1) = 0.\end{aligned} \tag{18}$$

[Note that in the notation of (16), $p = (1 + x)^2$, $q = 0$ and $\rho = 50(1 + x)$, so that the form of the differential equation in (18) satisfies the conditions $p > 0, q \geqslant 0, \rho > 0$ assumed on *W*160. We can therefore be sure that the theorems of *Unit 13* will apply to this system.]

The problem (18) cannot be solved exactly, but something can still be said about the shape of its eigenfunctions.



We know from the Oscillation Theorem [rather than from the Separation Theorem as is stated] that the kth eigenfunction of the system (18) has exactly $k - 1$ zeros in $(0, 1)$. This enables us to sketch the first four eigenfunctions as shown in the left-hand half of the photograph above. [Clearly the sketch for X_4 is in error; we require $X_4(1) = 0$.] The pattern continues for later eigenfunctions. Using equations (1) we can deduce the corresponding water surface configurations $\eta_n(x, t_0)$, $n = 1, 2, \ldots$, at some fixed time $t = t_0$ for which $\eta_n(x, t)$ is not momentarily passing through its equilibrium position. The zeros of X_n correspond to the maxima or minima of $\eta_n(x, t_0)$. [This follows from the first of equations (1). We see from the second equation that the zeros of $\eta_n(x, t_0)$ correspond to the maxima or minima of $(1 + x)X_n$. It is not therefore true as stated that the zeros of $\eta_n(x, t_0)$ correspond to the turning points of X_n. There must however be a turning point of $(1 + x)X_n$ between any pair of consecutive zeros of X_n, so we may justifiably conclude that $\eta_n(x, t_0)$ has a zero between any two consecutive zeros of X_n. See Appendix for further details.] We obtain the shapes of water surfaces corresponding to the first four eigenfunctions of (18) as shown in the right-hand half of the photograph; the slope of $\eta_n(x, t_0)$ is zero at each end of the tank.

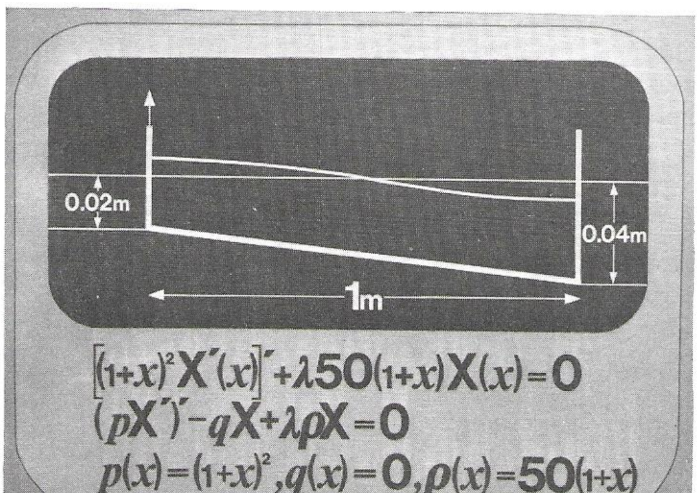
Knowledge of the eigenvalues allows us to calculate the period of the corresponding standing wave motions. Equation (6) has the solution

$$T(t) = C \sin [(\lambda g)^{\frac{1}{2}} t + c], \tag{19}$$

and this has period

$$P = 2\pi/\sqrt{\lambda g}. \tag{20}$$

The period of the standing wave corresponding to the nth eigenfunction of system (18) is therefore found by substituting the nth eigenvalue for λ in this equation. We now concentrate on estimating the lowest eigenvalue λ_1 which will provide a corresponding approximation to the period of the simplest standing wave.



The Monotonicity Theorem is used to place upper and lower bounds on λ_1. We see that in the interval $[0, 1]$ the functions $p(x) = (1 + x)^2$ and $\rho(x) = 50(1 + x)$ satisfy the inequalities

$$\begin{aligned} 1 &\leqslant p(x) \leqslant 4, \\ 100 &\geqslant \rho(x) \geqslant 50. \end{aligned} \tag{21}$$

The theorem then tells us that the nth eigenvalue of the system (18) is greater than the nth eigenvalue of (16) with $p \equiv 1, q \equiv 0, \rho \equiv 100$, and less than the nth eigenvalue of (16) with $p \equiv 4, q \equiv 0, \rho \equiv 50$, the same boundary conditions pertaining in each case. The first eigenvalue of (18) therefore lies between the first eigenvalues of the systems

$$\begin{aligned} &X'' + 100\lambda X = 0 \qquad 0 < x < 1, \\ &X(0) = X(1) = 0, \end{aligned} \tag{22}$$

and

$$4X'' + 50\lambda X = 0 \qquad 0 < x < 1,$$
$$X(0) = X(1) = 0. \tag{23}$$

These first eigenvalues are readily found to be $\pi^2/100$ and $4\pi^2/50$ respectively, so that

$$\frac{\pi^2}{100} < \lambda_1 < \frac{4\pi^2}{50}, \tag{24}$$

or

$$0.098 < \lambda_1 < 0.790. \tag{25}$$

Substituting these bounds into equation (20) we find

$$6.39\text{s} > P > 2.25\text{s}. \tag{26}$$

This is hardly a spectacularly accurate estimate for the period but it's only a first attempt.

Conclusion

Ralph Smith states that the procedure needed to improve the bounds found for the period P of the simplest standing wave is one of trial and error. He previews Radio Programme 7 which is devoted to finding better bounds on P. The problem is approached using the minimum principle

$$\lambda_1 \leqslant \frac{\int_0^1 [p(\phi')^2 + q\phi^2]dx}{\int_0^1 \rho\phi^2 dx}, \tag{27}$$

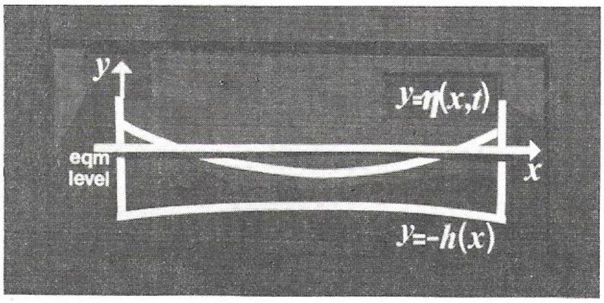
where $p(x) = (1 + x)^2$. $q(x) \equiv 0$, $\rho(x) = 50(1 + x)$, and ϕ is any continuous piecewise continuously differentiable function which vanishes at $x = 0$ and $x = 1$. The particular function ϕ used in the radio programme is

$$\phi(x) = x(1 - x) \qquad 0 \leqslant x \leqslant 1. \tag{28}$$

APPENDIX

Derivation of the Shallow Water Wave Equation

Consider a tank with constant longitudinal cross-section containing liquid. As before we take the x-axis to lie along the equilibrium level of the liquid. The surface profile of the liquid is given by $y = \eta(x, t)$ and the shape of the bottom of the tank by $y = -h(x)$.



We make the following initial assumptions:

(a) The motion of any particular fluid particle lies at all times in a plane parallel to that formed by the x- and y-axes, i.e. we consider the fluid motion to be two-dimensional.

(b) The fluid is incompressible, so that its density ρ is constant.

(c) The effects of viscosity are negligible.

(d) The vertical component of acceleration of the liquid particles has a negligible effect on the pressure; in other words the expression for the pressure $p(x, y, t)$ at the point (x, y) of the fluid at time t is taken to depend linearly on depth as in the hydrostatic case, so that

$$p(x, y, t) = \rho g[\eta(x, t) - y], \tag{29}$$

where g is the acceleration due to gravity.

(e) There exists at least one value of t for which the horizontal component u of the liquid particle velocity does not vary with depth.

Assumption (d) is basic to shallow water theory. Differentiating equation (29) with respect to x,

$$p_x = \rho g \eta_x, \tag{30}$$

from which we see that p_x, the pressure gradient along the tank, is independent of y. This pressure gradient is however (up to sign) the x-component of the force per unit volume, so we may deduce from Newton's Second Law that the horizontal component of particle acceleration, du/dt, is independent of depth. It follows from assumption (e) that u is independent of y at all times. This means that particles lying in any plane perpendicular to the x-axis will remain in a moving vertical plane as they sweep to and fro.

Consider the fluid flowing into the space between two fixed vertical planes P through x and P' through $x + \Delta x$. We can form an equation of conservation of mass by equating the net inflow rate of fluid into the region between the two planes with the rate of volume increase due to the vertical velocity of the surface.

The volume flow rate from left to right through P is $b[\eta(x, t) + h(x)]u(x, t)$, where b is the width of the tank. The volume flow rate from left to right through P' is $b[\eta(x + \Delta x, t) + h(x + \Delta x)]u(x + \Delta x, t)$, so that if Δx is small the net rate of inflow into the region between P and P' will be

$$b[\eta(x, t) + h(x)]u(x, t) - b[\eta(x + \Delta x, t) + h(x + \Delta x)]u(x + \Delta x, t)$$
$$\simeq -b[(\eta + h)u]_x \Delta x. \tag{31}$$

The vertical velocity of the surface at x is η_t, and the corresponding rate of change of volume between P and P' is approximately $\eta_t b \Delta x$. Equating this with the right-hand side of (31),

$$[(\eta + h)u]_x = -\eta_t. \tag{32}$$

We obtain a second equation relating u and η by writing down Newton's Second Law for motion in the x-direction. The particle acceleration in this direction is $du/dt = u_t + uu_x$ (compare with line 1 on page 15 of *Unit I*) and the force per unit volume is $-p_x$. Hence

$$\rho(u_t + uu_x) = -p_x. \tag{33}$$

Using equation (30) this becomes

$$u_t + uu_x = -g\eta_x. \tag{34}$$

We have two equations connecting u and η. The additional assumption is now made that u and η, together with their derivatives, are small quantities whose squares and products can be ignored in comparison with linear terms. When this simplification is made equations (32) and (34) yield equations (1) of these Broadcast Notes,

$$\begin{aligned} u_t &= -g\eta_x, \\ (hu)_x &= -\eta_t. \end{aligned} \tag{1}$$

Differentiating the first of these with respect to x, the second with respect to t, and equating the two expressions for $-\eta_{xt}$ gives the shallow water wave equation

$$(hu)_{xx} = \frac{1}{g}u_{tt}. \tag{2}$$

A more comprehensive derivation of equations (1) can be found on pages 23, 24 of *Water Waves* by J. J. Stoker (Interscience, 1957).

Derivation of $\eta_n(x, t)$ From $u_n(x, t) = X_n(x)\,T_n(t)$

It seems intuitively reasonable that the function $\eta_n(x, t)$ corresponding to the standing wave solution $u_n(x, t) = X_n(x)T_n(t)$ should itself represent a standing wave (and hence be a product of a function of x and a function of t). It is not however mathematically obvious that this need be so, and it is clear that such a function would not be the unique solution to equations (1).

If $\eta_n^{(1)}$ and $\eta_n^{(2)}$ are assumed to be two distinct solutions of (1) then $\bar{\eta}_n = \eta_n^{(1)} - \eta_n^{(2)}$ must satisfy $(\bar{\eta}_n)_x = (\bar{\eta}_n)_t = 0$, whence any two solutions may differ by a constant. We therefore proceed to construct a separated solution for $\eta_n(x, t)$ on the assumption that there is one, and then show that the constant which may be added to this solution to form a further solution of (1) must in fact be zero from other considerations.

Suppose $u_n(x, t) = X_n(x)T_n(t)$, where X_n is the nth eigenfunction of the system (5) and T_n the corresponding solution of equation (6). Suppose further that $\eta_n(x, t) = F_n(x)G_n(t)$, and that equations (1) are satisfied by $u = u_n$, $\eta = \eta_n$. We have

$$\begin{aligned} (hX_n)'T_n &= -F_nG_n', \\ X_nT_n' &= -gF_n'G_n. \end{aligned} \tag{35}$$

Separation constants α_n, β_n can therefore be found such that

$$\begin{aligned} \frac{(hX_n)'}{F_n} &= -\frac{G_n'}{T_n} = \alpha_n, \\ \frac{T_n'}{gG_n} &= -\frac{F_n'}{X_n} = \beta_n, \end{aligned} \tag{36}$$

and these in turn give

$$\begin{aligned} F_n &= (hX_n)'/\alpha_n, \\ G_n &= T_n'/g\beta_n, \end{aligned} \tag{37}$$

so that

$$\eta_n = (hX_n)'T_n'/g\alpha_n\beta_n. \tag{38}$$

Differentiating the first of equations (37) and substituting for F_n' in the second of equations (36) we see that

$$(hX_n)'' + \alpha_n\beta_nX_n = 0, \tag{39}$$

and it follows from (5) that $\alpha_n\beta_n = \lambda_n$. Hence

$$\eta_n = (hX_n)'T_n'/\lambda_ng. \tag{40}$$

As has been mentioned a constant may be added to the right-hand side of (40) and the resulting expression will still satisfy equations (1). Suppose $\bar{\eta}_n(x, t) = \eta_n(x, t) + c$. The volume of liquid in the tank must always be the same and consequently equal to the volume in the equilibrium state. For all t therefore,

$$\int_0^l \bar{\eta}_n(x, t)dx = 0 \tag{41}$$

if $\bar{\eta}_n$ is to be a physically satisfactory expression for the surface profile. Since

$$\int_0^l \bar{\eta}_n(x, t)dx = [h(l)X_n(l) - h(0)X_n(0)]T_n'(t)/\lambda_ng, \tag{42}$$

which vanishes by the boundary conditions of the system (5), we see that $c = 0$ and that $\bar{\eta}_n = \eta_n$. Equation (40) therefore gives the unique expression for $\eta_n(x, t)$ corresponding to $u_n(x, t) = X_n(x)T_n(t)$. In particular it gives (13) as the surface profile corresponding to the velocity function (12), since in this case $\lambda_1 = \pi^2 H/l^2$.

To Show That the Zeros of $u_n(x, t_0)$ and $\eta_n(x, t_0)$ Interleave

We wish to show that exactly one zero of $\eta_n(x, t_0)$ lies between each pair of consecutive zeros of $u_n(x, t_0)$. From equation (40) this is equivalent to showing that exactly one zero of $(hX_n)'$ lies between each pair of adjacent zeros of X_n. Since all the zeros of X_n are zeros of hX_n there must be at least one turning value of hX_n between any two consecutive zeros of X_n, i.e. there must be at least one zero of $(hX_n)'$.

X_n must be of constant sign between any two of its adjacent zeros. Suppose without loss of generality that $X_n > 0$ between two such zeros. It follows from (5) together with the inequality $\lambda_n > 0$ that

$$(hX_n)'' < 0 \tag{43}$$

between the two zeros in question. Any turning point of hX_n in this interval must therefore be a maximum, and since two maxima of a continuous function are necessarily separated by a minimum there can only be one such maximum. There can therefore only be one zero of $(hX_n)'$ between the two zeros of X_n.

Note incidentally that the inequality (43) and the reverse inequality for the case when $X_n < 0$ provide more information with which to sketch the graphs of the X_n.

M321 R7

RADIO PROGRAMME 7

Broadcast Notes

SHALLOW WATER WAVES

The investigation of Television Programme 3 into shallow water waves is continued. After a recapitulation of material in TV3, improvements on the bounds previously calculated for the period of the simplest standing wave motion in a rectangular tank with sloping bottom are sought. Successively more restrictive bounds are found by

(a) applying the minimum principle to the first eigenvalue of the Sturm-Liouville system which arises from the shallow water wave equation,

(b) looking at a different Sturm-Liouville system which has the same eigenvalues as the system in (a), and bounding the first eigenvalue using the Monotonicity Theorem.

(c) applying the minimum principle to the first eigenvalue of the system in (b).

In each case the bounds on the period can readily be calculated from those on the first eigenvalue.

The presenters are Ralph Smith and Dominic Jordan, and the introduction is by Ralph Smith. Reference is made to the following notes in the programme.

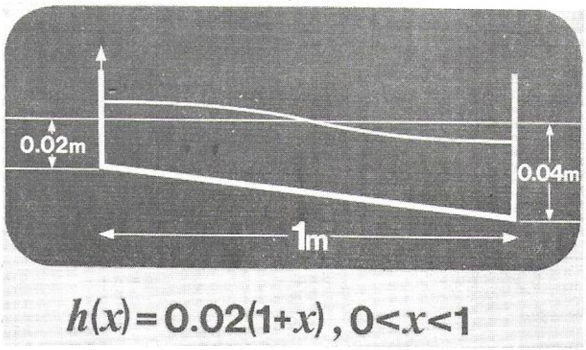


Fig. 1

Putting $u(x, t) = X(x)T(t)$ into the shallow water wave equation

$$(hu)_{xx} = \frac{1}{g}u_{tt},$$

where $h(x)$ is given in this case by the expression in the photograph above, and using the boundary conditions $u(0, t) = u(1, t) = 0$, we find that

$$[(1 + x)X]'' + \lambda 50X = 0, \qquad (1)$$

$$X(0) = X(1) = 0, \qquad (2)$$

$$T'' + \lambda gT = 0. \qquad (3)$$

After equation (1) has been put into self-adjoint form we have

$$[(1 + x)^2X']' + \lambda 50(1 + x)X = 0, \qquad (4)$$

$$X(0) = X(1) = 0, \qquad (5)$$

$$T'' + \lambda gT = 0, \qquad (6)$$

and the first two of these equations form a Sturm-Liouville system of the type considered in Chapter VII of *W*. Corresponding to each eigenvalue λ_n there is an eigenfunction $X_n(x)$ and a standing wave $u_n(x, t) = X_n(x)T_n(t)$. The period of this standing wave is the period of $T_n(t)$, which is a solution of equation (6) found when $\lambda = \lambda_n$, namely

$$T_n(t) = C \sin [(\sqrt{\lambda_n g}t) + c]. \tag{7}$$

[The constants C and c should have subscript n in this context but it has been omitted.] This has period

$$P_n = 2\pi/\sqrt{\lambda_n g}. \tag{8}$$

The complete standing wave corresponding to λ_n has the form

$$u_n(x, t) = X_n(x) . C \sin [(\sqrt{\lambda_n g}t) + c] \tag{9}$$

We now concentrate on the first eigenvalue λ_1. This corresponds to the simplest standing wave $u_1(x, t)$ which has period $P_1 = 2\pi/\sqrt{\lambda_1 g}$. It was found in Television Programme 3 that

$$0.098 < \lambda_1 < 0.790, \tag{10}$$

and consequently that

$$6.39\text{s} > P_1 > 2.25\text{s} \tag{11}$$

(see inequalities (25), (26) of TV3 Broadcast Notes).

We next try and improve these bounds using the minimum principle

$$\lambda_1 \leqslant \frac{\int_0^1 [p(\phi')^2 + q\phi^2]dx}{\int_0^1 \rho\phi^2 dx}, \tag{12}$$

where in this case $p(x) = (1 + x)^2$, $q(x) \equiv 0$ and $\rho(x) = 50(1 + x)$. ϕ is any continuous piecewise continuously differentiable function which vanishes at $x = 0$ and $x = 1$. Equality holds if and only if $\phi = X_1$. We might suppose that $\phi(x) = \sin \pi x$ would give a close approximation to λ_1 since this was found to be the first eigenfunction in the case of constant depth (put $l = n = 1$ in equation (9) of the TV3 Broadcast Notes). The integration involved is a little lengthy however so we choose instead the function

$$\phi(x) = x(1 - x) \qquad 0 \leqslant x \leqslant 1, \tag{13}$$

which has a similar graph to that of $\sin \pi x$ but ensures that all the integrands in (12) will be polynomials.

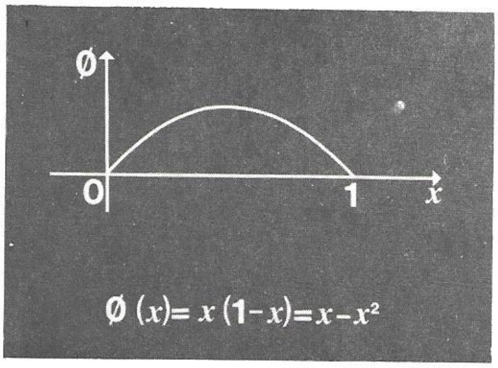


Fig. 2

We get

$$\lambda_1 \leqslant \frac{\int_0^1 (1 + x)^2(1 - 2x)^2 dx}{\int_0^1 50(1 + x)(x - x^2)^2 dx}, \tag{14}$$

which gives

$$\lambda_1 \leqslant 0.320. \tag{15}$$

Putting this together with (10) we see that

$$0.098 < \lambda_1 \leqslant 0.320,$$

and hence

$$6.39\text{s} > P_1 \geqslant 3.55\text{s}. \tag{16}$$

[Putting $\phi(x) = \sin \pi x$ produces an upper bound of $(14\pi^2 + 3)/450 \simeq 0.314$ for λ_1 and a corresponding lower bound of 3.58s for P_1, so hardly any improvement is gained using this expression for $\phi(x)$ instead of that in equation (13).]

The problem can be reformulated by putting

$$Y = (1 + x)X, \tag{17}$$

and substituting into equations (1) and (2):

$$Y'' + \frac{50\lambda}{1 + x} Y = 0, \tag{18}$$

$$Y(0) = Y(1) = 0. \tag{19}$$

The eigenvalues of the system (18), (19) are the same as those of the system (1), (2), for if $X_n(x)$ is the nth eigenfunction of (1), (2) it will satisfy equation (1) with $\lambda = \lambda_n$. $Y_n(x) = (1 + x)X_n(x)$ will then satisfy (18) with $\lambda = \lambda_n$, and $Y_n(0) = X_n(0) = 0$, $Y_n(1) = 2X_n(1) = 0$. Equation (18) is moreover already in self-adjoint form. Equation (3) remains unchanged by the transformation (17):

$$T'' + \lambda g T = 0. \tag{20}$$

Comparing equation (18) with the standard form

$$(pX')' - qX + \lambda \rho X = 0, \tag{21}$$

we see that in this case $p(x) \equiv 1$, $q(x) \equiv 0$ and $\rho(x) = 50/(1 + x)$. Since $25 \leqslant \rho(x) \leqslant 50$ on the interval $[0, 1]$, we know by the Monotonicity Theorem that λ_1 must lie between the first eigenvalues of the systems formed by adding to the boundary conditions (19) the differential equations

$$Y'' + 50\lambda Y = 0, \tag{22}$$

$$Y'' + 25\lambda Y = 0. \tag{23}$$

The first eigenvalues of these systems are $\pi^2/50$ and $\pi^2/25$ respectively, so that

$$0.197 < \lambda_1 < 0.395. \tag{24}$$

Combining this with our previous best estimate (the inequalities prior to (16)) we have

$$0.197 < \lambda_1 \leqslant 0.320, \tag{25}$$

or

$$4.52\text{s} > P_1 \geqslant 3.55\text{s}.$$

The minimum principle for the Sturm-Liouville system (18), (19) is

$$\lambda_1 \leqslant \frac{\int_0^1 (\phi')^2 dx}{\int_0^1 \frac{50}{1 + x} \phi^2 dx}, \tag{26}$$

and using $\phi(x) = x(1 - x)$ once more,

$$\lambda_1 \leqslant \frac{\int_0^1 (1 - 2x)^2 dx}{\int_0^1 \frac{50(x - x^2)^2}{1 + x} dx}, \tag{27}$$

which works out to give

$$\lambda_1 \leqslant 0.295. \tag{28}$$

Comparing this with (25),

$$0.197 < \lambda_1 \leqslant 0.295,$$

so that

$$4.52\text{s} > P_1 \geqslant 3.69\text{s}.$$

Another way of estimating λ_1 is suggested by SAQ 23(c) of *Unit 13*, where it is shown that

$$\lim_{k \to \infty} \frac{\lambda_k}{k^2} = \frac{\pi^2}{\left[\int_\alpha^\beta \rho^{\frac{1}{2}}(x)dx\right]^2}, \tag{29}$$

where λ_k is the kth eigenfunction of the problem

$$u'' + \lambda \rho u = 0 \qquad \alpha < x < \beta,$$

$$u(\alpha) = u(\beta) = 0.$$

The expression on the right-hand side of (29) is a reasonable estimate for λ_k/k^2 even when k is not large provided that ρ is not too irregular. Putting $\alpha = 0, \beta = 1$ and (from equation (18)) $\rho(x) = 50(1 + x)^{-1}$, we arrive at the value 0.288 for the right-hand side of (29), which may be taken as an approximation to λ_1.

It is in fact possible to find λ_1 explicitly using the methods of *Unit 14*. It takes the approximate value 0.290, so that the upper bound of (28) and the value derived from (29) are seen to be quite good estimates. The corresponding value of P_1 is 3.73 s. This compares with values for a flat-bottomed tank of length 1 m and water depths 2 cm, 3 cm, 4 cm of 4.57 s, 3.69 s and 3.20 s respectively.

M321 TV4

TELEVISION PROGRAMME 4

Broadcast Notes

REVISION

This programme could loosely be described as adopting an algorithmic approach to the solution of a partial differential equation together with its associated initial and/or boundary conditions. The intention is to review the different types of equation which have arisen during the course, to look at the allowable initial/boundary conditions associated with each type, and to enumerate the possible analytic and numerical methods of solution available in each case. The programme is introduced by Daniel Lunn, and the second presenter is Peter Thomas.

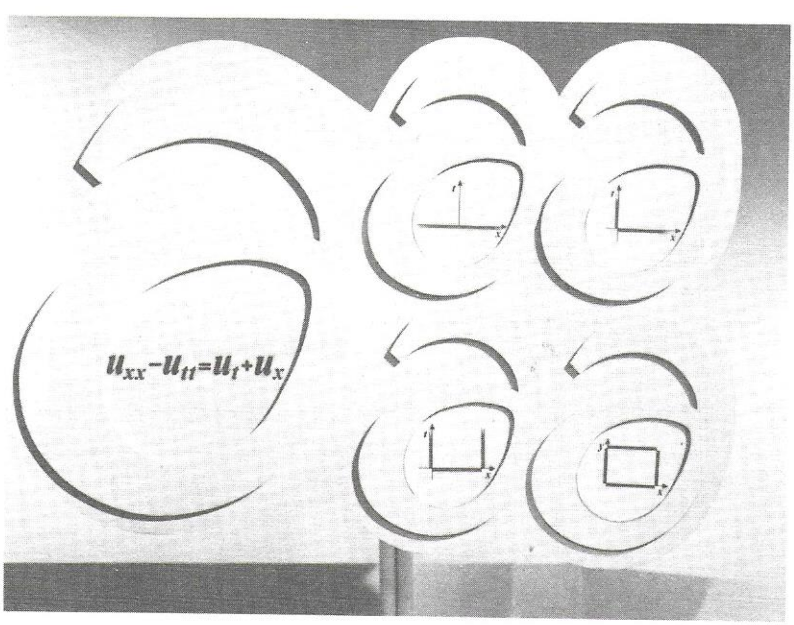
Categorizations

The first part of the programme makes periodic reference to the equation

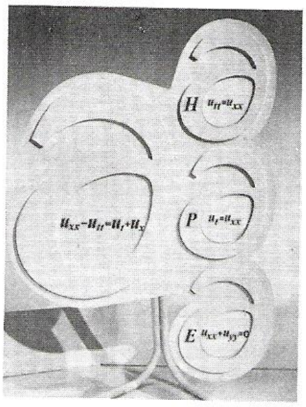
$$u_{xx} - u_{tt} = u_t + u_x \qquad (1)$$

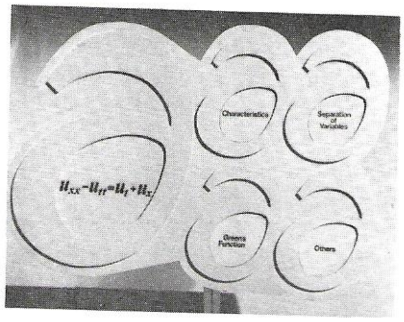
(where subscripts again denote partial differentiation with respect to the subscripted variable), and after a solution domain and certain initial conditions have also been specified the resulting problem is solved.

An initial and/or boundary value problem consists of both a partial differential equation, whose solution is required in some given domain, and a set of conditions to be satisfied by the solution on the boundary of this domain. Three broad categories of initial/boundary conditions are considered in the course, namely pure initial conditions, mixed initial-boundary conditions and pure boundary conditions; these may be symbolically represented as shown in the photograph. [The situation at top right is stated to represent a fourth distinct type of condition and a second kind of pure initial condition. Initial conditions of this type have not really been investigated in the course and do not in any case differ substantially from those represented by the diagram at top left.] The boundaries in the cases represented by the two lower diagrams might be at infinity, in which case the associated conditions would be expected to hold in the appropriate limits.

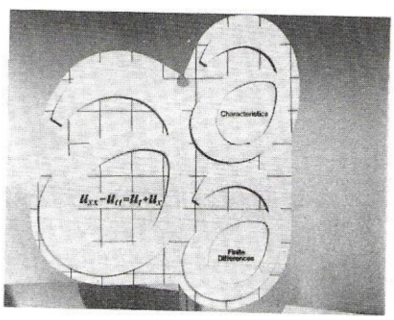


As a first step towards solving equation (1) we seek to classify it as hyperbolic, parabolic or elliptic, and then look for a method of solution. The three principal analytic methods mentioned in the course are those of characteristics, separation of variables, and Green's functions (other non-numerical techniques include the finite Fourier transform but that is not considered here).





We could also attempt a numerical solution if an analytic one is not forthcoming. Any information gleaned from the analysis, such as the likely effect of any discontinuity in the initial or boundary data, will be helpful when we try to solve the problem numerically. We may choose to use either the numerical method of characteristics or a finite-difference scheme, but having made this choice we must still decide how best to use them.



A solution domain and initial conditions are added to equation (1) in order to form a properly posed problem:

$$\left.\begin{aligned} &u_{xx} - u_{tt} = u_t + u_x && x \in R, \quad t > 0, \\ &u(x, 0) = xe^{\frac{1}{2}x} && x \in R, \\ &u_t(x, 0) = e^{\frac{1}{2}x}(1 - \tfrac{1}{2}x) && x \in R. \end{aligned}\right\} \qquad (2)$$

Equation (1) is classified by investigating the sign of the quantity $B^2 - 4AC$, where A, B, C are the coefficients of the u_{tt}, u_{xt} and u_{xx} terms respectively. Since in this case $A = -1$, $B = 0$ and $C = 1$, we have $B^2 - 4AC = 4 > 0$, so that the equation is hyperbolic.

Hyperbolic

We know that in general hyperbolic equations can occur in properly posed problems of initial condition or initial-boundary condition type, but not in pure boundary value problems. We can choose between two possible analytic methods of solution, the method of characteristics or separation of variables. The initial and/or boundary conditions usually determine which choice to make in order to derive the analytic solution in its simplest form. Characteristics are most suitable in pure initial value problems (we could also use the Fourier transform method introduced in *M201 31*, but that has not been considered in this course). In initial-boundary value problems

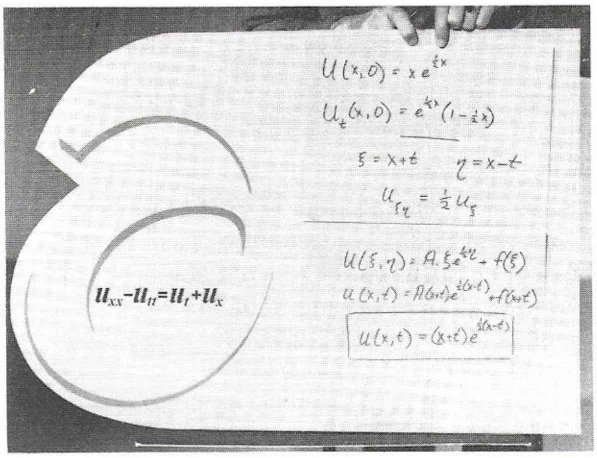
both characteristics and separation of variables are applicable but the latter method is usually preferred since it tends to reduce the problem to a Sturm–Liouville system. A solution to the problem may then be constructed by using the eigenfunctions of such a system.

The problem given by equations (2) is a pure initial value problem, so the method of characteristics is used to solve it. The viewer is expected to supply the details of the subsequent calculation, but a rough outline is given. The characteristic coordinates in this case are found to be

$$\xi = x + t, \eta = x - t, \tag{3}$$

and after transformation into the ξ, η coordinates equation (1) becomes

$$u_{\xi\eta} = \tfrac{1}{2}u_{\xi}. \tag{4}$$



The general solution of this equation is

$$u(\xi, \eta) = f(\xi)e^{\frac{1}{2}\eta} + g(\eta), \tag{5}$$

where f, g are arbitrary differentiable functions. [The general solution of equation (4) is given in the programme by the expression shown in the photograph above. Seasoned M321 students will have no difficulty in identifying the deliberate mistakes!]

Transferring back to x, t coordinates,

$$u(x, t) = f(x + t)e^{\frac{1}{2}(x-t)} + g(x - t). \tag{6}$$

In order to satisfy the initial conditions given in (2) we must choose $f(\xi) = \xi + c$, $g(\eta) = -ce^{\frac{1}{2}\eta}$, where c is an arbitrary real number. Hence the unique solution to the problem is

$$u(x, t) = (x + t)e^{\frac{1}{2}(x-t)}. \tag{7}$$

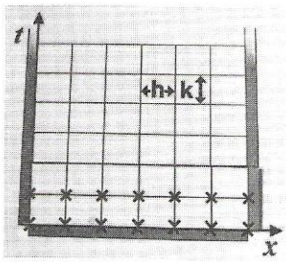
The problem (2) would not have been so simple to solve if an additional term, $+u$, had been added to the right-hand side of the partial differential equation. A term $+\frac{1}{4}u$ would appear on the right-hand side of the canonical form (4) [the factor $\frac{1}{4}$ is omitted in the programme], and it is not clear how the resulting equation can be solved analytically.

It can however be solved numerically, using either the method of characteristics or a finite-difference scheme. The first of these appears to be the better option in this case since the equations of the characteristics are already known and are simple. In general this method is to be preferred because it closely reflects a known analytic property of hyperbolic equations, namely that the solution is propagated along the characteristics or is otherwise governed by their geometrical disposition; in particular discontinuities in the initial or boundary data are propagated along the characteristics. Discontinuities in the solution cannot be so easily accommodated when using a finite-difference scheme. The numerical method of characteristics is described in section 2.3 of *Unit 2*.

Parabolic

We next consider parabolic equations, typified by the diffusion equation $u_t = u_{xx}$. These may occur in pure initial value problems or initial-boundary value problems [although the programme only acknowledges the second possibility] and the usual analytic method of solution is separation of variables [the finite Fourier transform may be applied in some cases, and this is essentially a Green's function technique; the method of *M201 31* would again be used for a pure initial value problem]. Separation of variables leads to two ordinary differential equations, one of which forms part of a system whose eigenvalues are to be found exactly or else estimated using the methods of *Unit 13*.

The only available numerical method for initial-boundary value problems involving parabolic equations is that of finite differences. The partial differential equation and associated conditions are replaced by finite-difference equations and the solution domain is covered with a rectangular mesh. The solution of the finite-difference scheme at the intersections of the mesh is sought. It is calculated on each successive time level using values previously computed for the preceding level, the process being started off with a set of known values on the initial line $t = 0$. Since the finite-difference scheme is only an approximation to the original problem, small errors (local truncation errors) are introduced into the calculation at each mesh point. We wish our scheme to be stable so that these errors will not accumulate. We also require that the accuracy of the finite-difference solution should improve as the mesh spacings h and k are decreased, and that the solution should tend to the true solution of the partial differential equation as $h, k \rightsquigarrow 0$ (which is the definition of convergence). For this to happen the local truncation errors must tend to zero as $h, k \rightsquigarrow 0$, i.e. the scheme must be consistent. In fact we know that, for the class of equations considered in the course, stability and consistency together imply convergence, so only the first two of these properties need be established.



Elliptic

There remain problems involving elliptic equations, such as Poisson's or Laplace's equations. In order to be properly posed such problems must be of pure boundary condition type. The possible methods of solution include separation of variables and Green's function techniques. The former is used wherever possible because it is manipulatively easier. Laplace's equation $u_{xx} + u_{yy} = 0$, being homogeneous, separates straight away, but Poisson's equation $u_{xx} + u_{yy} = \rho$ usually presents a tougher problem, since the ρ on the right-hand side could be any function of x and y. Even if it is not possible to separate Poisson's equation directly it may be that separation can take place after transformation to new coordinates, or after changing the dependent variable. For example the equation

$$u_{xx} + u_{yy} = 1 \tag{8}$$

is not immediately separable, but if we define $v(x, y)$ by the equation

$$u(x, y) = v(x, y) + \tfrac{1}{2}x^2, \tag{9}$$

we find that

$$v_{xx} + v_{yy} = 0, \tag{10}$$

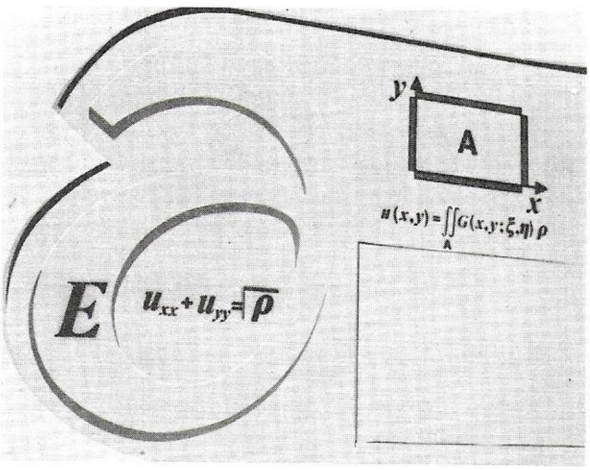
which is Laplace's equation once more.

If no transformation will enable us to use separation of variables we can fall back on Green's function techniques. If Poisson's equation

$$u_{xx} + u_{yy} = \rho \qquad (x, y) \in A \tag{11}$$

is part of a pure boundary value problem, the Green's function for the system, $G(x, y; \xi, \eta)$, will satisfy the equation

$$G_{xx} + G_{yy} = 0 \qquad (x, y) \neq (\xi, \eta) \tag{12}$$



together with appropriate conditions on the boundary of A. The solution to the problem will be given as

$$u(x, y) = -\int\int_A G(x, y; \xi, \eta)\rho(\xi, \eta)d\xi\, d\eta. \tag{13}$$

[This equation appears in the programme without a minus sign. This is a matter of convention, but the convention adopted in M321 (i.e. the precise way in which G has been defined there) necessitates the presence of a minus sign.] It is often difficult to find G, and even if G is found the integral in (13) may defy attempts to work it out. In such cases we might turn once more to numerical methods.

The finite-difference method is again employed for problems containing elliptic equations. The region covered by the mesh is closed on this occasion, and the partial differential equation and associated conditions are therefore replaced by the algebraic problem of solving a set of simultaneous equations, one for each mesh point. We have to consider whether the solution of the algebraic problem approximates the solution of the partial differential equation; we also need to know how to solve the algebraic problem. The first of these points is satisfactorily resolved if the solution of the algebraic problem tends to the solution of the original boundary value problem as the mesh spacings tend to zero (which is convergence once again). The only method for solving the algebraic problem which has been discussed in the course is simple iteration (the Gauss–Seidel or SOR methods for example).

M321 CN(A)
COURSE NOTES

Course Notes

ERRATA IN SET BOOKS

The following errata have been observed in the set books. These lists include reference to sections not contained in course reading passages.

Errata in *W*

W2, lines −5, −3: the expression for the force should read

$$\frac{T(s_2,t)\dfrac{\partial x}{\partial s}(s_2,t)}{\left[\sqrt{\left(\dfrac{\partial x}{\partial s}\right)^2+\left(\dfrac{\partial y}{\partial s}\right)^2}\right]_{s=s_2}} - \frac{T(s_1,t)\dfrac{\partial x}{\partial s}(s_1,t)}{\left[\sqrt{\left(\dfrac{\partial x}{\partial s}\right)^2+\left(\dfrac{\partial y}{\partial s}\right)^2}\right]_{s=s_1}}$$

W4, line 13: delete the full stop
W14, line −14: should read

$$u(0,t)=\tfrac{1}{2}[f(ct)+f(-ct)]+\frac{1}{2c}\int_{-ct}^{ct} g(\bar{x})\,d\bar{x}=0$$

W26, line 1: should read

$$\frac{1}{4c^2}\int_{\eta}^{\xi}\int_{\bar{\eta}}^{\xi} F\left(\frac{\bar{\xi}+\bar{\eta}}{2},\frac{\bar{\xi}-\bar{\eta}}{2c}\right)d\bar{\xi}\,d\bar{\eta}=\cdots$$

W27, line 8: should read

$$+\frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}}\left[\int_{-\frac{3}{4}+\bar{t}}^{0}(-\bar{t}\sin^2\pi\bar{x})\,d\bar{x}+\cdots\right.$$

W33, line 2: for v^1, v^2 read $v^{(1)}, v^{(2)}$
W34, line −15: for $L_1[\bar{u}]=f_1$ read $L_1[\bar{u}]=\bar{f}_1$
W35, exercise 5: should read

Show that the problem

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0 \quad \text{for } 0<x<l, t>0,$$

$$u(0,t)=f_1(t)$$

$$\frac{\partial u}{\partial x}(l,t)=f_2(t)$$

$$u(x,0)=f_3(x) \quad \text{for } 0\leqslant x\leqslant l$$

$$\frac{\partial u}{\partial t}(x,0)=f_4(x) \quad \text{for } 0\leqslant x\leqslant l$$

has at most one solution.

W36, line −5: for f_1, f_2, read f, g,
W53, line −13: The third integral in this equation should read

$$+\iint_D |\text{grad }u|^2\,dx\,dy$$

W53, line −11: for membrance *read* membrane
W55, line 9: for somehwere *read* somewhere
W74, line 16: for $f \equiv 0$ is the only function *read* functions f which are zero at all but a finite number of points are the only functions
W74, line 17: for its *read* their
W104, line −2: for U read u
W126, exercise 4: for $f(x)$ *read* $-f(x)$
W133, line 3: delete uniformly in ϕ
W138, line 1: for $0 < x < A$ *read* $0 < y < A$
W138, line 8: for $\int_0^\pi \int_0^\pi$ *read* $\int_0^A \int_0^\pi$

W142, line 15: should read

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y)^2 dx\, dy = \frac{\pi}{2} \int_{-\pi}^{\pi} a_0^2 dy + \pi \sum_1^\infty \int_{-\pi}^{\pi} [a_n^2 + b_n^2]\, dy$$

W167, line −6: for right-hand *read* left-hand
W172, line −6: for $\alpha < x \leqslant \bar{\alpha}$ *read* $\alpha \leqslant x \leqslant \bar{\alpha}$
W174, line 1: for $q\phi^2 dx)$ *read* $q\phi^2)\, dx$
W175, line 9: It is stated that "We can prove a monotonicity theorem like that given above for this problem". This statement is misleading. Monotonicity holds with respect to α, p, q, ρ and b, but not with respect to β.
W179, line 16: insert $\}$ *before* $=$
W180, line 4: for (41.1) *read* (40.1)
W182, line 11: for $\sin n\theta$ *read* $\sin m\theta$
W186, equation (42.2): for $c_{km}(t)''$ *read* $c_{km}''(t)$
W186, line −1: for C_{n0} *read* C_{k0}
W190, line −1: should read

$$\frac{(-2)^{k+1} c_{k+1}}{(-2)^k c_k} = 1 - \frac{k + 1 + \lambda}{(k+1)^2} > 1 - \frac{1}{k + 1 - \lambda} = \frac{k - \lambda}{k + 1 - \lambda}$$

W191, line 2: for nc_n *read* $n\, c_n$
W194, equation (44.3): for ψ *read* ϕ *and for* $\sin \theta\, d\theta\, d\phi$ *read* $\sin \bar{\theta}\, d\bar{\theta}\, d\bar{\phi}$
W377, line −3: insert) *after* k
W385, line 3: for (83.2) *read* (82.3)
W387, line 20: for $[\sin \bar{x} + \sin \bar{x}\,($ *read* $[\sin \bar{x} + \sin^2 \bar{x}($
W399, line 5: for $\left(\frac{\partial x}{\partial x}\right)^2$ *read* $\left(\frac{\partial x}{\partial s}\right)^2$

W402, solution to section 7, exercise 2: should read

2. The part of the strip $0 \leqslant x \leqslant \frac{\pi}{4}$, where

   $$0 \leqslant t \leqslant 2 - \left|\log \cos x - \log \cos \frac{\pi}{8}\right|.$$

W402, solution to section 7, exercise 3, line 2: should read
$0 \leqslant t \leqslant 3 - |\tan^{-1} x - \tan^{-1} \tfrac{1}{4}|$.
W402, solution to section 7, exercise 4, line 2: should read
$t \geqslant |\tan^{-1} x - \tan^{-1} \tfrac{1}{2}|$.

W404, solution to section 10, exercise 2, line 2: for $\sqrt{\alpha/Dz}$ *read* $\sqrt{\alpha/D}z$

W404, solution to section 10, exercise 2, lines 2 to 5: for
If the transformation $\xi_i \ldots$ and $AD > 0$
read
If the transformation $\xi_i = \alpha_i x + \beta_i y + \gamma_i z$, $i = 1, 2, 3$, reduces the three-dimensional operator to standard form, choose unit vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ which are perpendicular to each other and to $\boldsymbol{\gamma}$, and show that the transformation $\bar{\xi}_1 = \boldsymbol{\mu} \cdot \boldsymbol{\alpha} x + \boldsymbol{\mu} \cdot \boldsymbol{\beta} y$, $\bar{\xi}_2 = \boldsymbol{\nu} \cdot \boldsymbol{\alpha} x + \boldsymbol{\nu} \cdot \boldsymbol{\beta} y$ reduces the two-dimensional operator to standard form.
W404, solution to section 10, exercise 3: for $\nabla^2 u$ *read* $\sigma \nabla^2 u$

W417, solution to section 30, exercise 1, line 3: for $\log (r^2$ *read* $\log [r^2$
W418, solution to section 32, exercise 3, line 2: should read

$$\frac{\sin(n-\frac{1}{2})x\sin(2m-1)y\sinh\sqrt{[(n-\frac{1}{2})^2+(2m-1)^2+1]}(1-z)}{(2m-1)\sqrt{[(n-\frac{1}{2})^2+(2m-1)^2+1]}\cosh\sqrt{[(n-\frac{1}{2})^2+(2m-1)^2+1]}}$$

W422, solution to section 41, exercise 2, line 2: for $\dfrac{\sin c\sqrt{\lambda_k^{(m)}}t}{c\sqrt{\lambda_k^{(m)}}t}$ *read* $\dfrac{\sin c\sqrt{\lambda_k^{(m)}}t}{c\sqrt{\lambda_k^{(m)}}}$

W423, line −12: for $\alpha \sinh \xi, \sin \eta$ *read* $\alpha \sinh \xi \sin \eta$
W423, lines −7 to −1: should read

$$Y'' + [\lambda - \alpha^2\omega^2\cos^2\eta]Y = 0 \quad \text{for } 0 < \eta < 2\pi,$$
$$Y(2\pi) - Y(0) = 0$$
$$Y'(2\pi) - Y'(0) = 0$$
$$X'' + [\alpha^2\omega^2\cosh^2\xi - \lambda]X = 0 \quad \text{for } 0 < \xi < \tanh^{-1}\left(\frac{b}{a}\right),$$
$$X'(0) = X\left(\tanh^{-1}\left(\frac{b}{a}\right)\right) = 0 \quad \text{if } Y(2\pi - \eta) = Y(\eta),$$
$$X(0) = X\left(\tanh^{-1}\left(\frac{b}{a}\right)\right) = 0 \quad \text{if } Y(2\pi - \eta) = -Y(\eta).$$

The η-problem can be solved to give eigenvalues $\lambda_1(\omega), \lambda_2(\omega), \ldots$ as functions of ω.
Then the ξ-problem is used to determine a discrete set of values of ω.
W424, solution to section 44, exercise 4: should read

$$4.\ u(r,\theta,\phi,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} e^{-\mu_k^{(n+(1/2))}t} r^{-1/2} J_{n+(1/2)}(\sqrt{\mu_k^{(n+(1/2))}}r)$$
$$\left\{\tfrac{1}{2}a_{nk0}P_n(\cos\theta) + \sum_{m=1}^{\infty}(a_{nkm}\cos m\phi + b_{nkm}\sin m\phi)P_n^m(\cos\theta)\right\}$$

where $\sqrt{\mu_k^{(n+(1/2))}}\, J'_{n+(1/2)}(\sqrt{\mu_k^{(n+(1/2))}}) - \frac{1}{2}J_{n+(1/2)}(\sqrt{\mu_k^{(n+(1/2))}}) = 0,$
$k = 1, 2, \ldots$

$$a_{nkm} = \frac{(2n+1)(n-m)!\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^1 f(r,\theta,\phi)r^{-1/2}J_{n+(1/2)}(\sqrt{\mu_k^{(n+(1/2))}}r)P_n^m(\cos\theta)r^2\sin\theta\cos m\phi\,dr\,d\theta\,d\phi}{2\pi(n+m)!\displaystyle\int_0^1 J_{n+(1/2)}(\sqrt{\mu_k^{(n+(1/2))}})^2 r\,dr}$$

W440, lines −2, −1: should read

$$|u(1,1) - u_2(1,1)| \leqslant \tfrac{1}{2}(\cosh 2 - 3)$$
$$\cong 0.381$$

Errata in S

S10, lines 4, 5: for $\dfrac{\partial x}{\partial X}$ *read* $\dfrac{dx}{dX}$

S10, line −7: for (1.7) and (1.8) *read* (1.8) and (1.10)
S18, line −9: for $(0 \leqslant x \leqslant 1)$ *read* $(0 < x < 1)$
S26, line −4: for $u_i^{(n)}$ *read* $u_{i+1}^{(n)}$
S29, line 13: for $2(+r)$ *read* $2(1+r)$
S34, line 8: for $u_{0\,j}$ *read* $u_{0,j}$
S34, line −8: for $(0 \leqslant x \leqslant 1)$ *read* $(0 < x < 1)$
S54, lines −3, −2: for i read n
S59, lines −12, −11: for modulus of the maximum *read* maximum modulus of the
S62, line −3: should read
since the e's and v's are known and the v's are independent.
S66, lines −5, −4: for a_{ss} *read* $a_{s,s}$
S66, line −1: should read

$$\lambda_i = a_{s,1}\left(\frac{v_1}{v_s}\right) + a_{s,2}\left(\frac{v_2}{v_s}\right) + \cdots + a_{s,s} + \cdots + a_{s,n}\left(\frac{v_n}{v_s}\right).$$

S69, lines 4 to *7:* should read

Application of Brauer's theorem to this matrix, with $a_{s,s}$ taking the possible values $1 - 2r$, $1 - 2r(1 + h_1\delta x)$, $1 - 2r(1 + h_2\delta x)$, and $P_s = 2r$, shows that each of its eigenvalues λ lies within at least one of the discs
$|\lambda - \{1 - 2r\}| \leqslant 2r$, $|\lambda - \{1 - 2r(1 + h_1\delta x)\}| \leqslant 2r$, $|\lambda - \{1 - 2r(1 + h_2\delta x)\}| \leqslant 2r$.

S74, line 12: for mean-value *read* intermediate value
S79, line 11: for on *read* of
S98, line −4: delete u,

S122, line 6: for $\sum^{\infty}$ *read* $\sum_{n=1}^{\infty}$

S130, line −6: for M read M_2
S166, line 6: for 70.5 *read* 70.5°
S172, solution to exercise 16: It is unjustifiably assumed that the problem has diagonal symmetry in addition to the stated symmetries with respect to Ox and Oy. There should therefore be a fourth pivotal value considered, at the point $(0, \frac{1}{2})$, giving a fourth equation. The required results can still be deduced though care must be taken to ensure that the four equations are consistently ordered.

M321 CN(B)
COURSE NOTES

Course Notes

ERRATA IN COURSE UNITS

Errata in Unit 1

Page 7, line −3: It is stated that "The MASS of PQ is $\rho\Delta s$." Since any point of the string is considered to move only in the transverse direction (*page 6, lines −4, −3*) the mass of that portion of the string between P and Q must always be the same as the mass between points x and $x + \Delta x$ in the equilibrium position, namely $\rho\Delta x$. Although the length of this portion of string will certainly increase when in motion, its line density will decrease correspondingly as the string stretches. ρ is therefore the *equilibrium line density* of the string, not the dynamic line density.

This mistake has repercussions in the ensuing argument. *Lines −2, −1* on *page 7* are redundant. In *line 7* of *page 8* the "sec ψ" should be omitted. In *equation (1)* "$\cos^2 \psi$" should be "$\cos \psi$", and in *equation (2)* "$\cos^4 \psi$" should be "$\cos^3 \psi$". *Line −9* on *page 7* should read

$$\cos^3 \psi = \frac{1}{\sec^3 \psi} = \frac{1}{(1 + \tan^2 \psi)^{\frac{3}{2}}} = \frac{1}{[1 + (\partial y/\partial x)^2]^{\frac{3}{2}}}$$

and "$\cos^4 \psi$" should be "$\cos^3 \psi$" in *line −7*.

Erratum in Unit 2

Page 10, line −3: for $t \times 0$ *read* $t > 0$

Errata in Unit 3

Page 7, line −4: for **t** such **t** · **grad** ϕ *read* **t** such that **t** · **grad** ϕ
Page 8, lines −18 and −16: for line −10 *read* line −11
Page 17, line 6: there should be a factor $\frac{1}{2}$ before the integral sign
Page 22, line 2: insert (solution on p. 40) *underneath this line*
Page 38, line 1: the second integral should be preceded by a minus sign
Page 42, line 10: for v^* *read* $\mathbf{v}^*$

Errata in Unit 5

Pages 20, 21, Library Program $CNH321: The version of this program listed is in error and has subsequently been corrected and otherwise amended. Those interested are invited to obtain a listing of the revised program at their local terminal.
Page 28, line −3: for $T^*_{i,j} = kT_{i,j+1}$ *read* $T^*_{i,j+1} = kT_{i,j}$

Errata in Unit 7

Page 11, line −8: for means *read* mean
Page 15, sixth line after the diagram: for $v \sin \vartheta/\Delta\bar{t}$ *read* $v \tan \theta/\Delta\bar{t}$
Page 19, equation (19): for $e^{-\sigma 1|\bar{x}|}$ *read* $e^{-\sigma_1|\bar{x}|}$
Page 25, line 18: for ω *read* $z(x, t)$

Page 26, line 7: for $\frac{1}{c^2} + \frac{\partial\bar{y}}{\partial\bar{t}}$ *read* $\frac{1}{c^2}\frac{\partial^2\bar{y}}{\partial\bar{t}^2}$

Page 28, line 7: for $ct - x$ *read* $ct + x$